Implementation of different controllers in rotary inverted pendulum



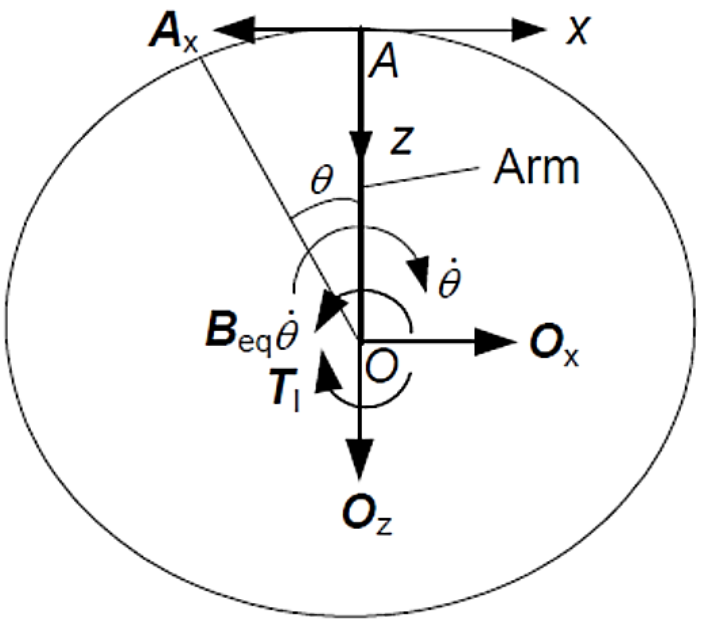
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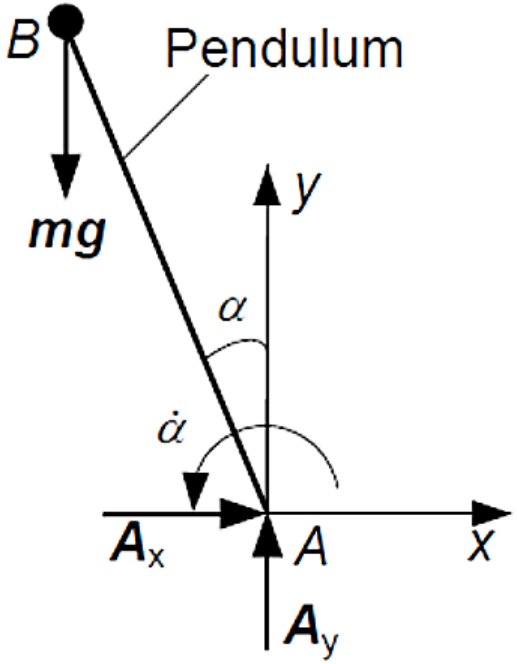
Guided By,
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Mathematical Modelling

Free body Diagram of Arm



Free Body Diagram of Pendulum



- Lagrange method is used to find the equations of motion of the system given by :

$$\frac{\partial^2 L}{\partial t \partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$$

- The variables q_i are called the generalized coordinates.

$$q(t)^T = \begin{bmatrix} \theta(t) & \alpha(t) \end{bmatrix}$$

- Qi are used to describe non-conservative forces applied to the system

$$Q_1 = \tau - B_r\dot{\theta}$$

$$Q_2 = -B_p\dot{\alpha}$$

- Where τ is the Torque applied , B_r and B_p are Coefficients of friction .
- The Lagrangian of the system is described as :

$$L = T - V$$

Where T is the total kinetic energy and V is the total Potential energy of the system.

Using Lagrangian formulation Equations of Motions are found to be :

$$\left(m_p L_r^2 + \frac{1}{4} m_p L_p^2 \cos(\alpha)^2 + J_r\right)\ddot{\theta} - \left(\frac{1}{2} m_p L_p L_r \cos(\alpha)\right)\ddot{\alpha}$$
$$+\left(\frac{1}{2} m_p L_p^2 \sin(\alpha)\cos(\alpha)\right)\dot{\theta}\dot{\alpha} + \left(\frac{1}{2} m_p L_p L_r \sin(\alpha)\right)\dot{\alpha}^2 = \tau - B_r\dot{\theta}$$

$$-\frac{1}{2} m_p L_p L_r \cos(\alpha)\ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right)\ddot{\alpha} - \frac{1}{4} m_p L_p^2 \cos(\alpha)\sin(\alpha)\dot{\theta}^2$$
$$-\frac{1}{2} m_p L_p g \sin(\alpha) = -B_p\dot{\alpha}$$

Where Torque applied ,

$$\tau = \frac{\eta_g K_g \eta_m k_t (V_m - K_g K - m\dot{\theta})}{R_m}$$

LINEARIZING

After linearizing at $\theta = 0°,\ \alpha = 0°,\ \dot{\theta} = 0,\ \dot{\alpha} = 0$ the equations of motion are found to be :

$$(m_p L_r^2 + J_r)\ddot{\theta} - \frac{1}{2} m_p L_p L_r \ddot{\alpha} = \tau - B_r \dot{\theta}$$

$$-\frac{1}{2} m_p L_p L_r \ddot{\theta} + \left(J_p + \frac{1}{4} m_p L_p^2\right)\ddot{\alpha} - \frac{1}{2} m_p L_p g \alpha = -B_p \dot{\alpha}$$

Simplifying the above equations by eliminating $\ddot{\theta}$ and $\ddot{\alpha}$

$$A = \frac{1}{J_T} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{1}{4} m_p^2 L_p^2 L_r g & -(J_p + \frac{1}{4} m_p L_p^2) B_r & -\frac{1}{2} m_p L_p L_r B_p \\ 0 & \frac{1}{2} m_p L_p g (J_r + m_p L^2 r) & \frac{1}{2} m_p L_p L_r B_r & -(J_r + m_p L_r) B - p \end{bmatrix}$$

$$B = \frac{1}{J_T}\begin{bmatrix} 0 \\ 0 \\ J_p + \frac{1}{4}m_p L_p^2 \\ \frac{1}{2}m_p L_p L_r \end{bmatrix}$$

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

Where , $x = \begin{bmatrix} \Theta \\ \alpha \\ \dot{\Theta} \\ \dot{\alpha} \end{bmatrix}$

State Space parameters

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 52.78 & -0.66 & -0.65 \\ 0 & 97.98 & -0.65 & -1.21 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 274.45 \\ 273.92 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Transfer functions:

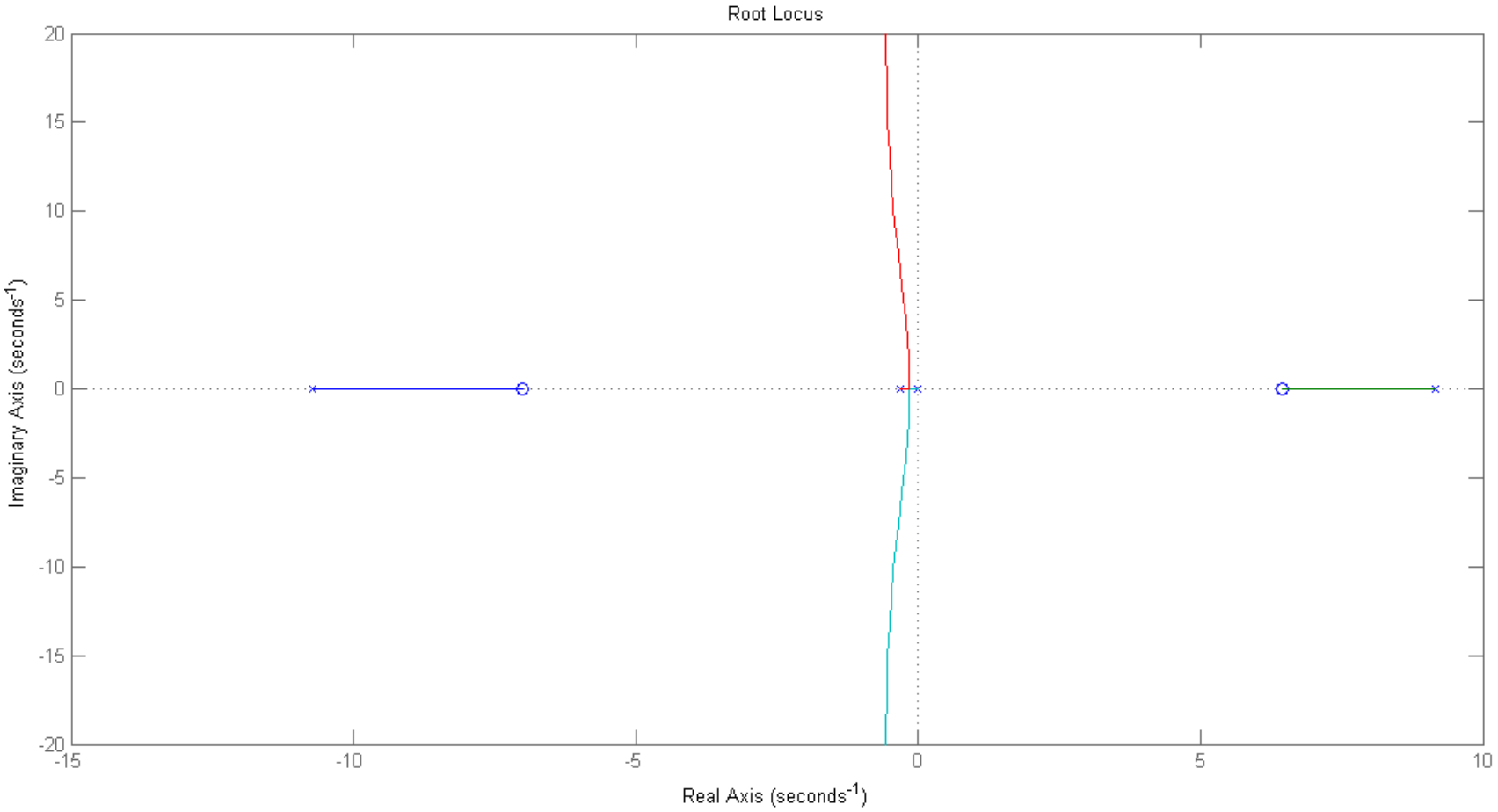
$$\frac{\Theta(s)}{V(s)} = \frac{274.5s^2 + 154s - 12440}{s^4 + 1.87s^3 - 97.61s^2 - 30.37s}$$

$$\frac{\alpha(s)}{V(s)} = \frac{273.9s}{s^3 + 1.87s^2 - 97.61s - 30.37}$$

Conventional Model for Control System Analysis

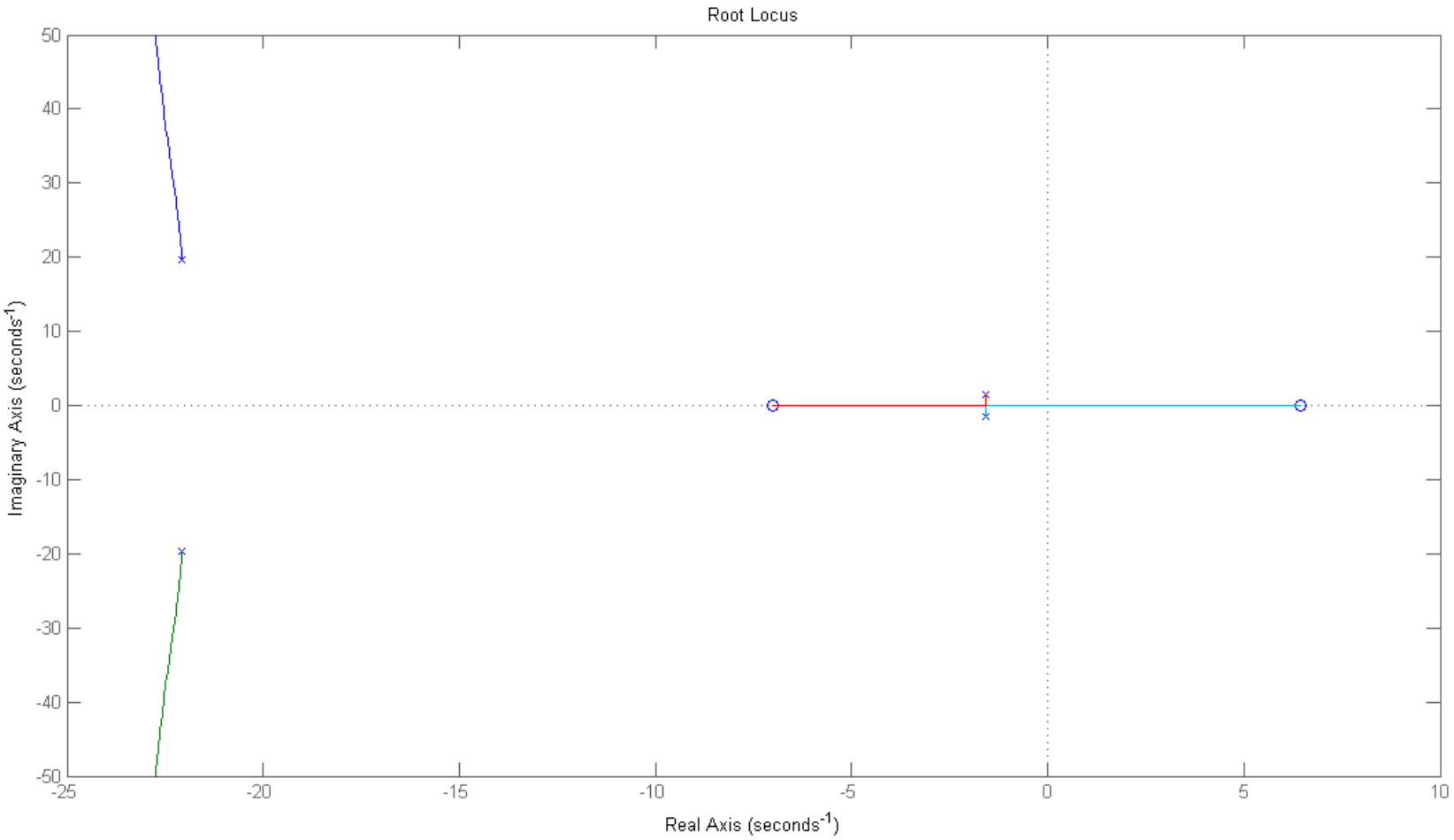
Root Locus of Uncompensated System

$$\frac{\Theta(s)}{V(s)}$$



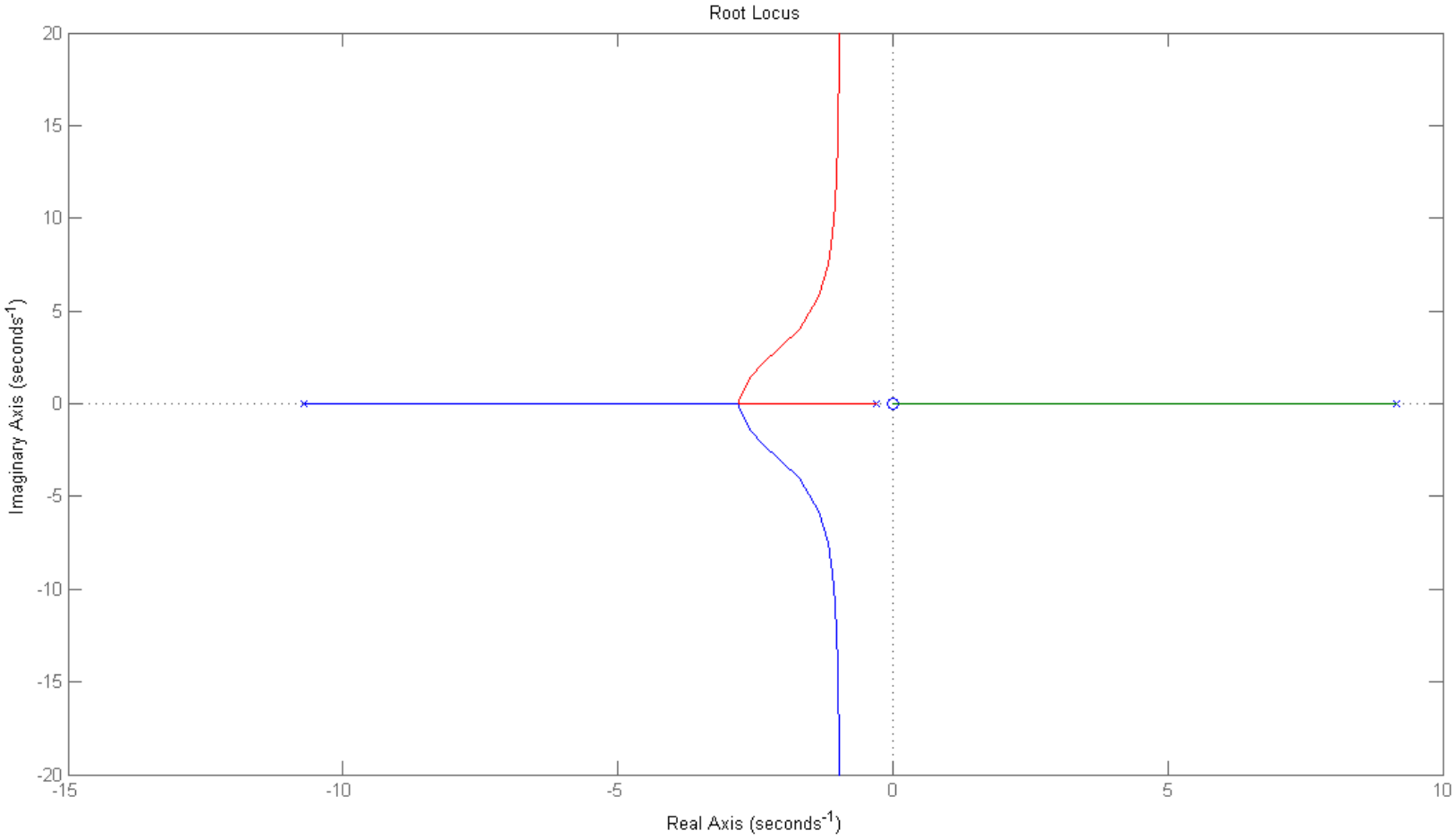
Root Locus of compensated System

$$\frac{\Theta(s)}{V(s)}$$

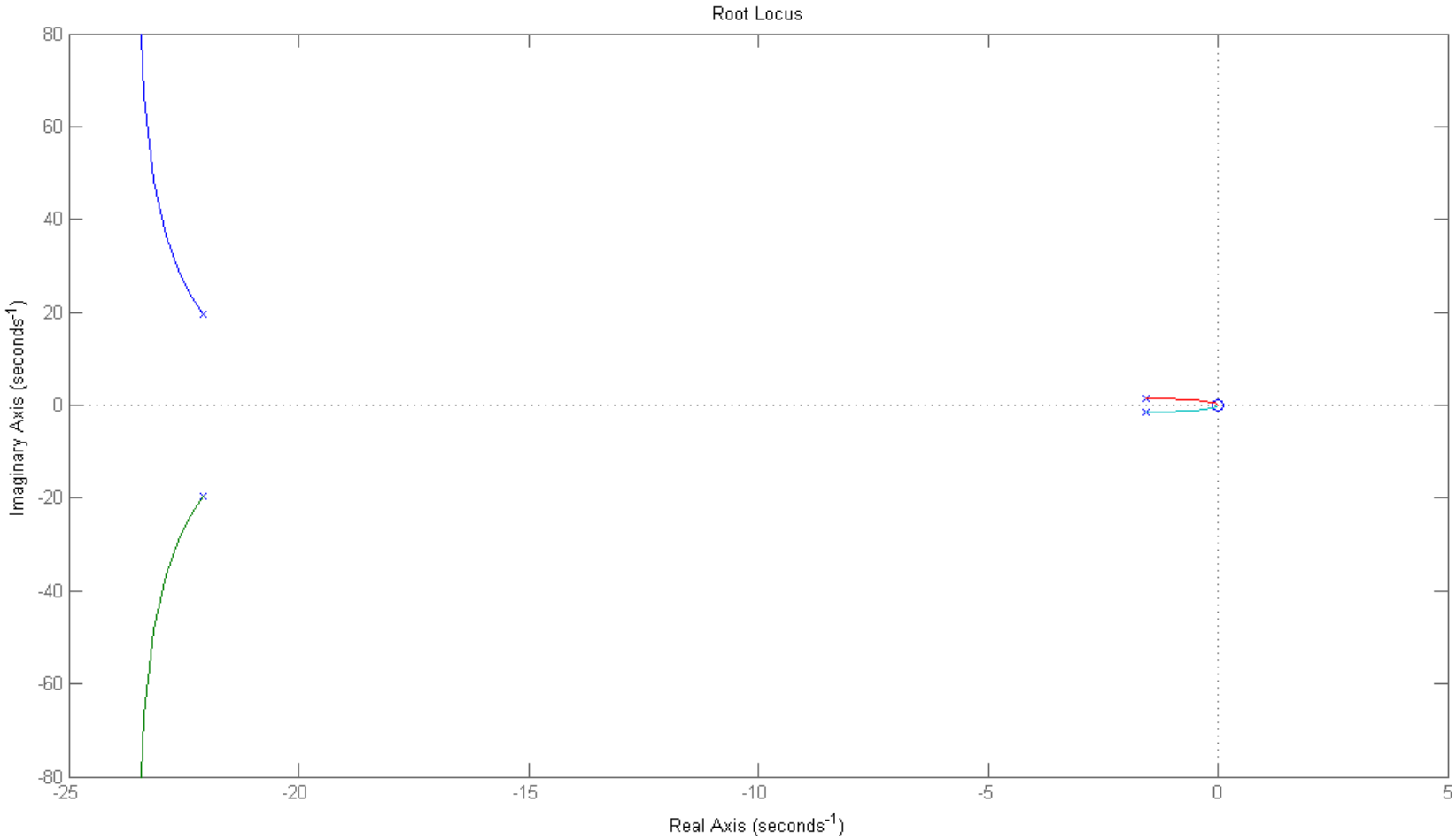


Root Locus of Uncompensated System

$$\frac{\alpha(s)}{V(s)}$$

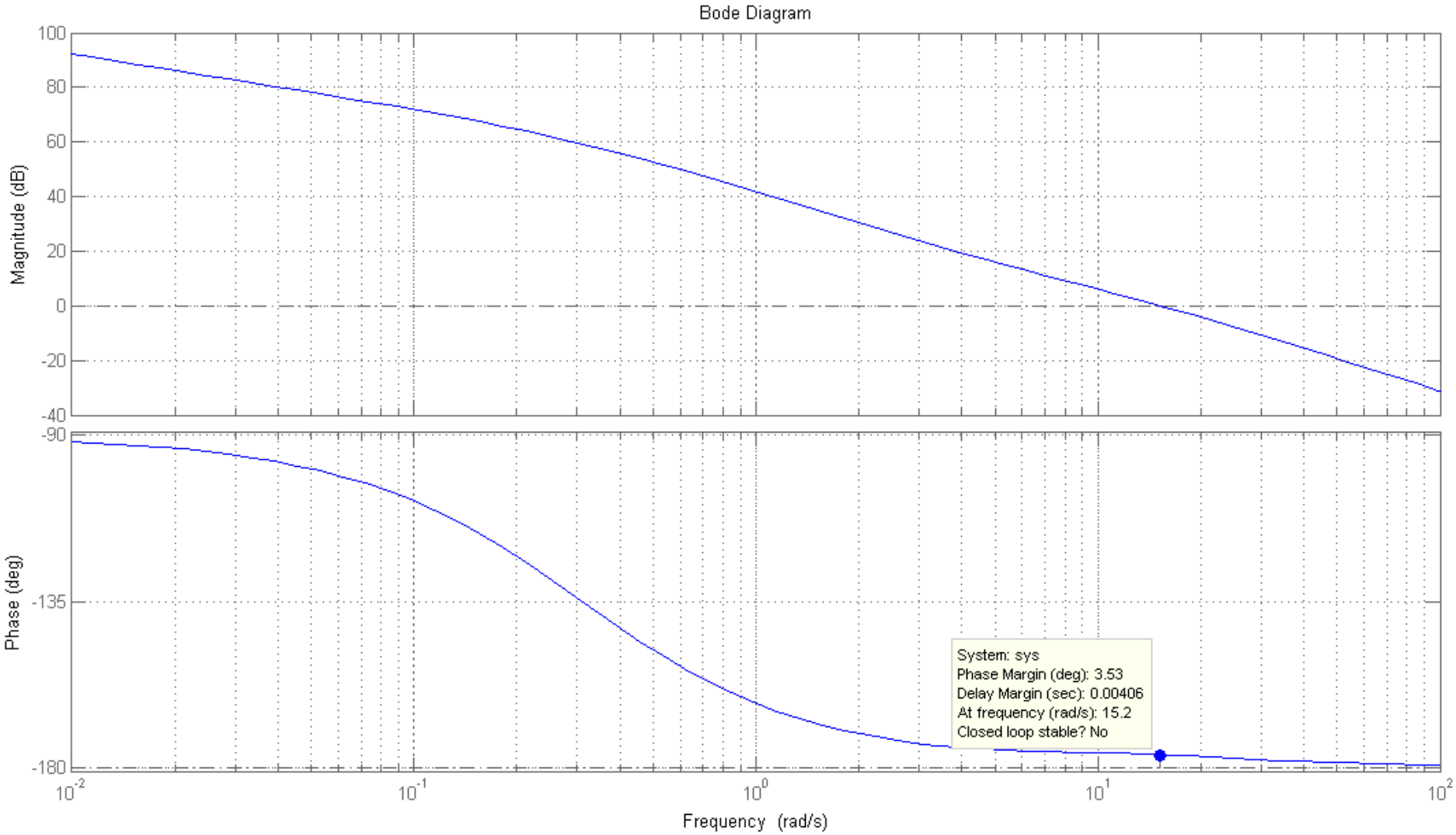


Root Locus of compensated System $\frac{\alpha(s)}{V(s)}$

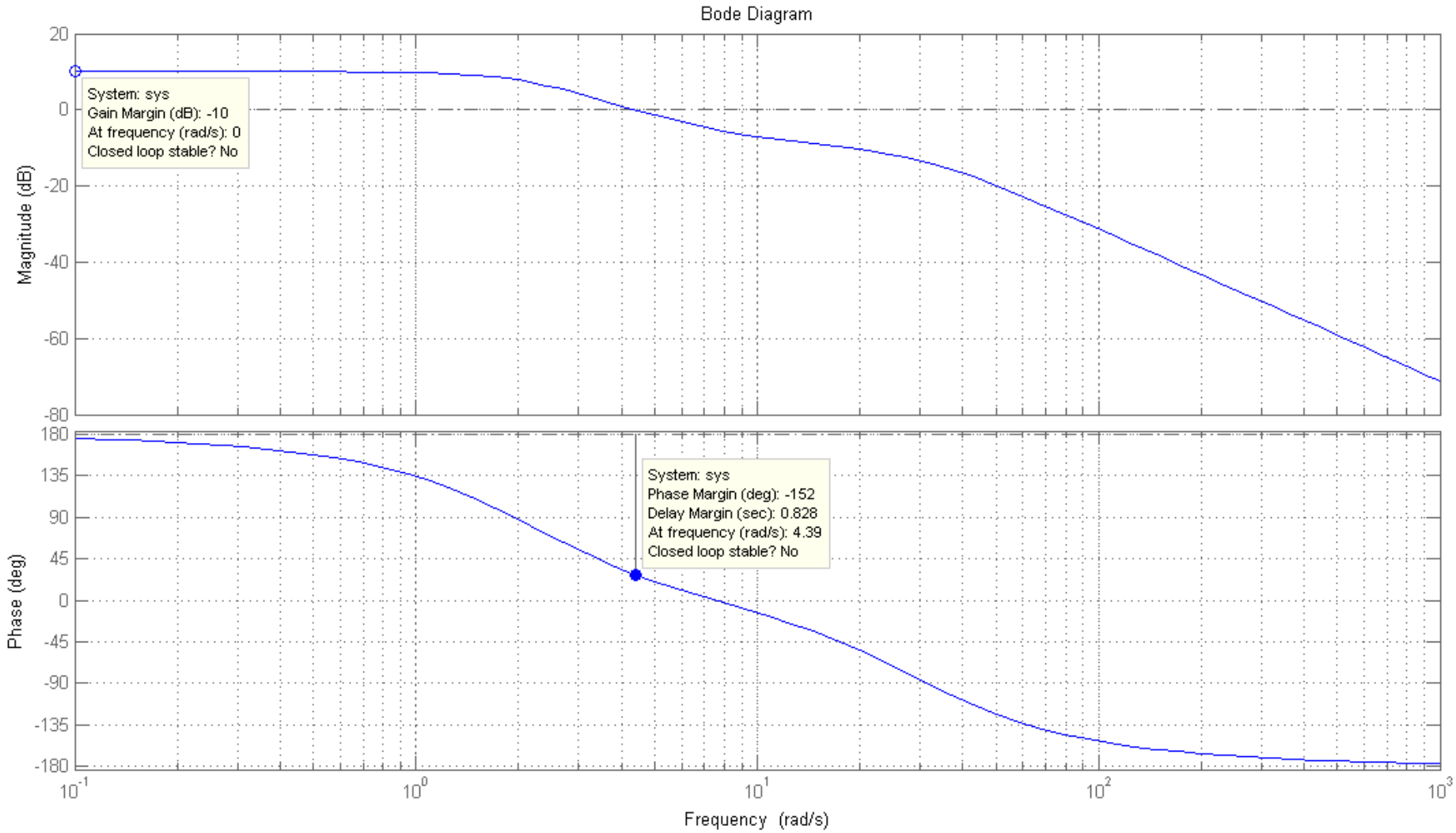


Frequency response

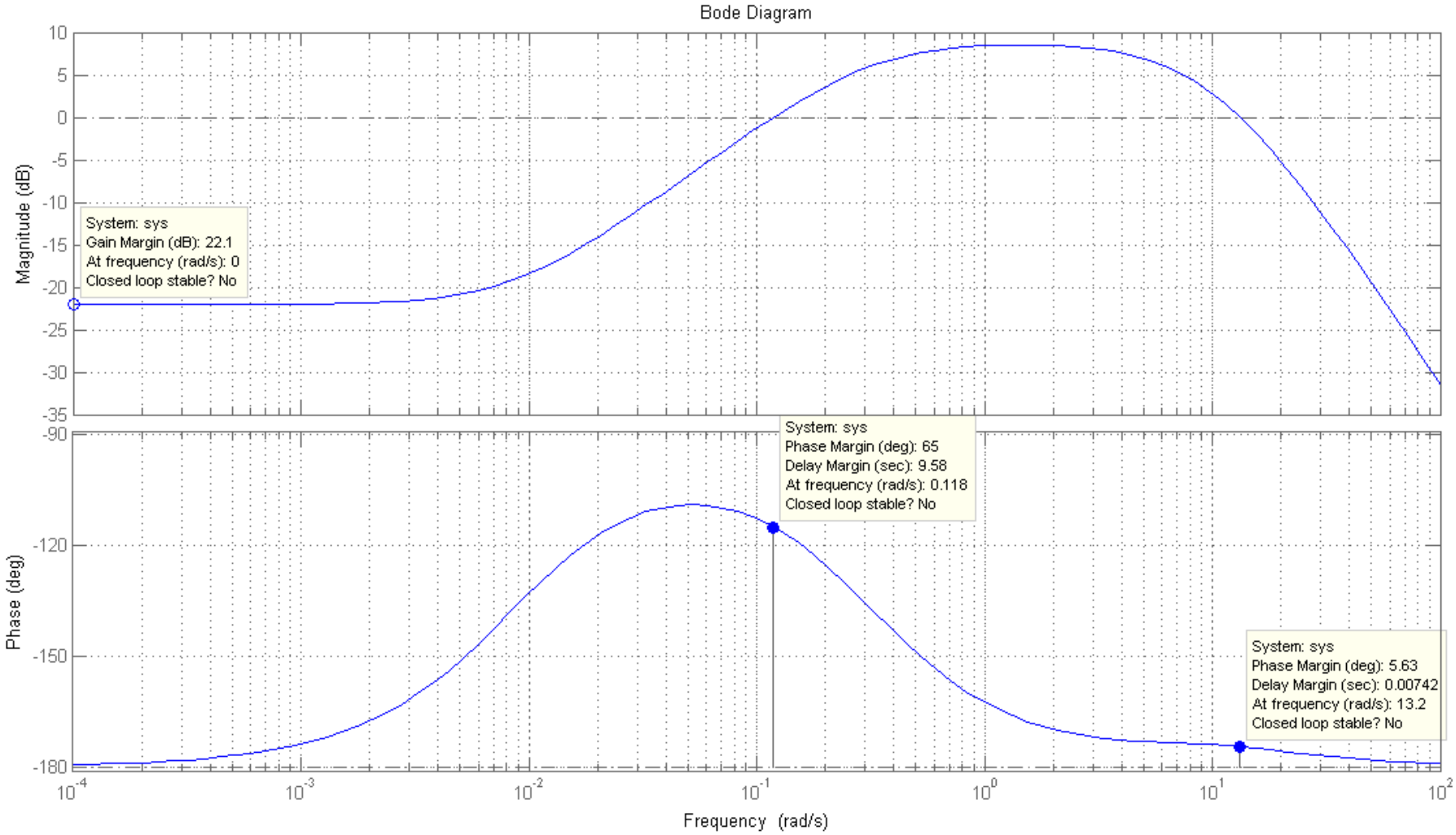
Bode Plot for θ uncompensated system



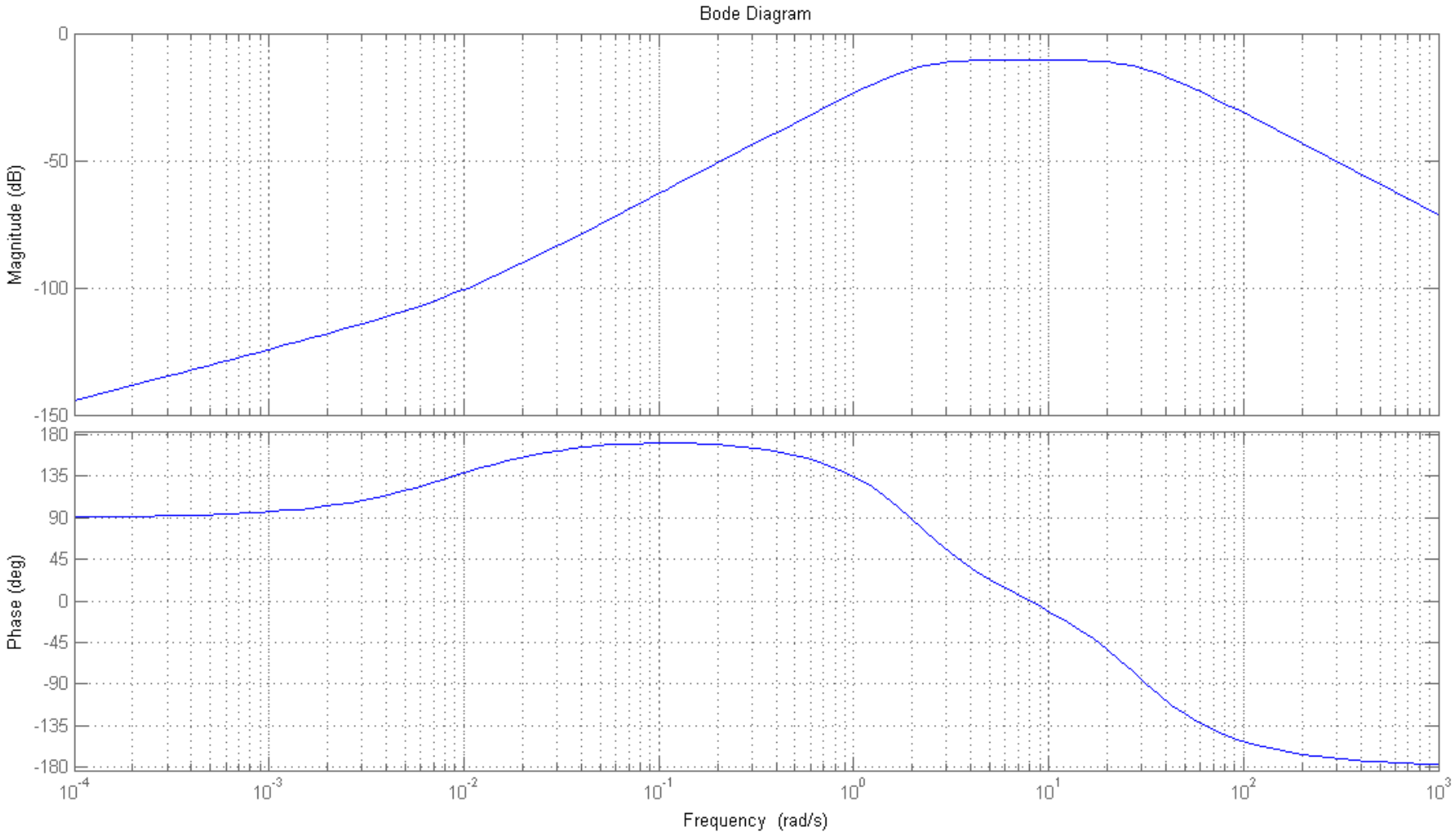
Bode Plot for θ compensated system



Bode Plot for α uncompensated system



Bode Plot for α compensated system



Modern Control system analysis

STATE FEEDBACK USING LQR OPTIMIZATION

Controllability

$C = [B\ AB\ A^2B\ A^3B]$

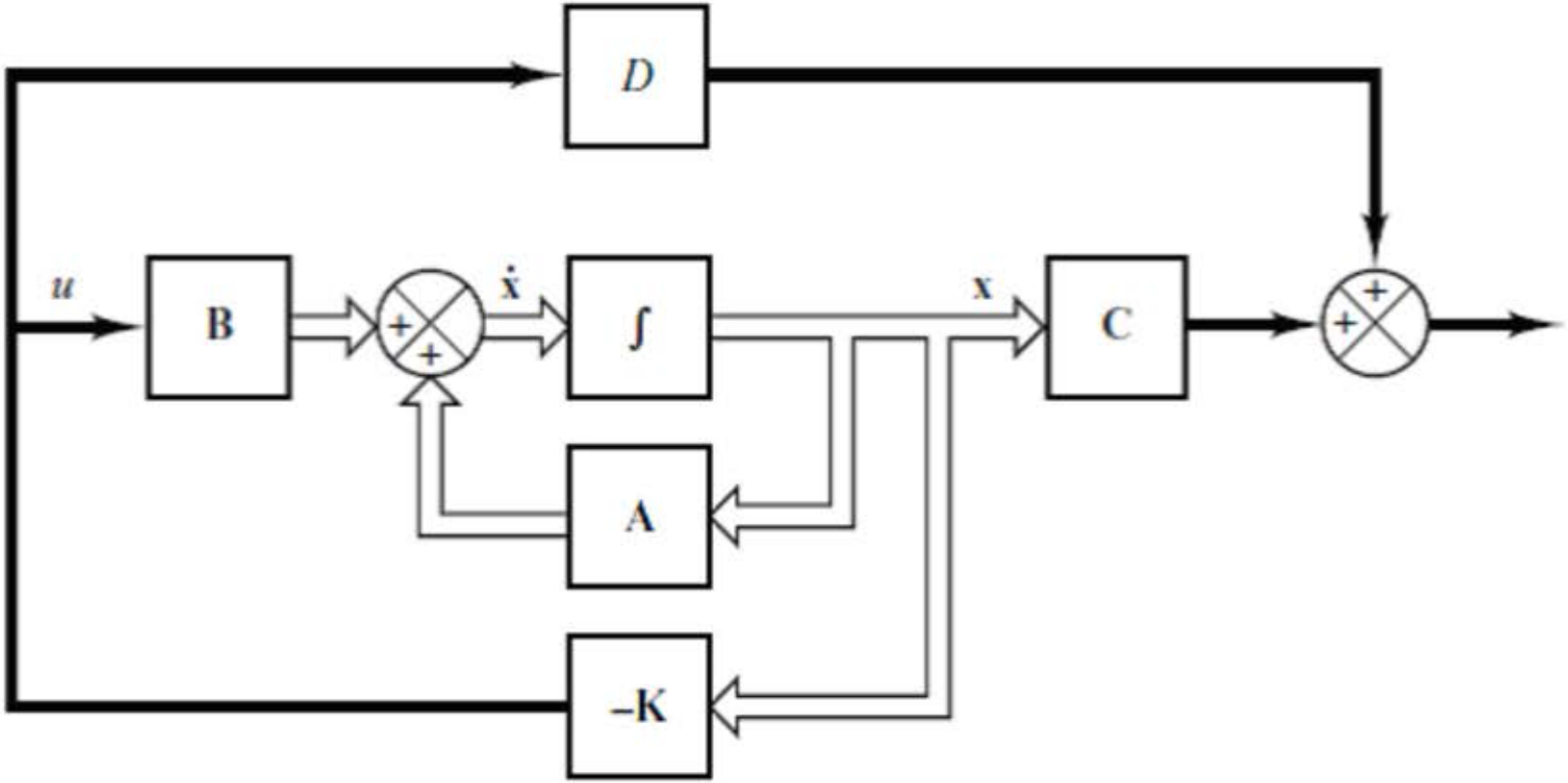
C is the Controllable Matrix

Rank of C is found to be 4 , hence the system is completely controllable.

State feedback using LQR optimization

- State Feedback Control | Pole-placement Method:
  - The system should be completely state controllable.
  - Poles of the closed-loop system may be placed at any desired locations by means of state feedback through an appropriate state feedback gain matrix.
  - Eigen values of original system state matrix A and state matrix system with state feedback A-BK are the same.

Block Diagram : LQR method



- The system is expressed in state variable form as

$$\dot{x} = Ax + Bu$$

- We assume that all the states are measurable. The state variable feedback control can be found from the expression :

$$u = -kx$$

- Using this control closed loop system becomes

$$\dot{x} = (A - Bk)x$$

Linear Quadratic Regulator (LQR) Optimization

- One of the main results of the theory of optimal control
- It is concerned with operating a system at the minimum cost.
- Cost function J expressed as the integral of a quadratic form in the state '*x*' plus a second quadratic form in the control '*u*'

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dx$$

- The matrices Q and R determine the relative importance of the error and the expenditure of the energy of control signal.

- The design procedure for finding the feedback gain ‘ *k* ’ for LQR :

1) Select the design parameter matrices Q and R.

2) Find P by solving the Algebra riccati equation :

$$A^T P + PA - PBR^{-1}B^T P + Q = 0$$

3) Find the state feedback matrix ‘ *k* ’ using :

$$k = R^{-1}B^T P$$

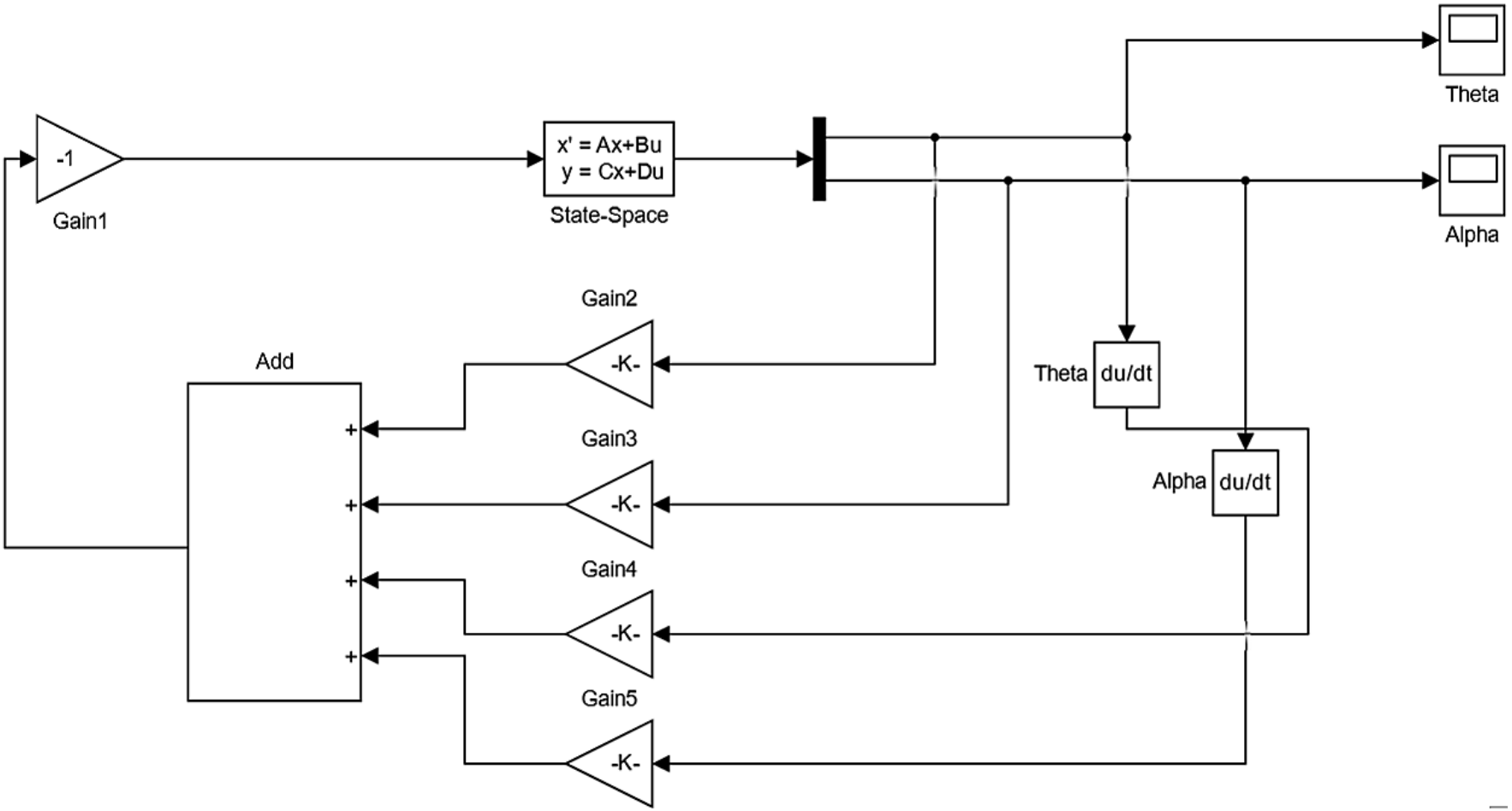
- After thorough Hit and Trial method Q and R are found to be :

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R = 10$$

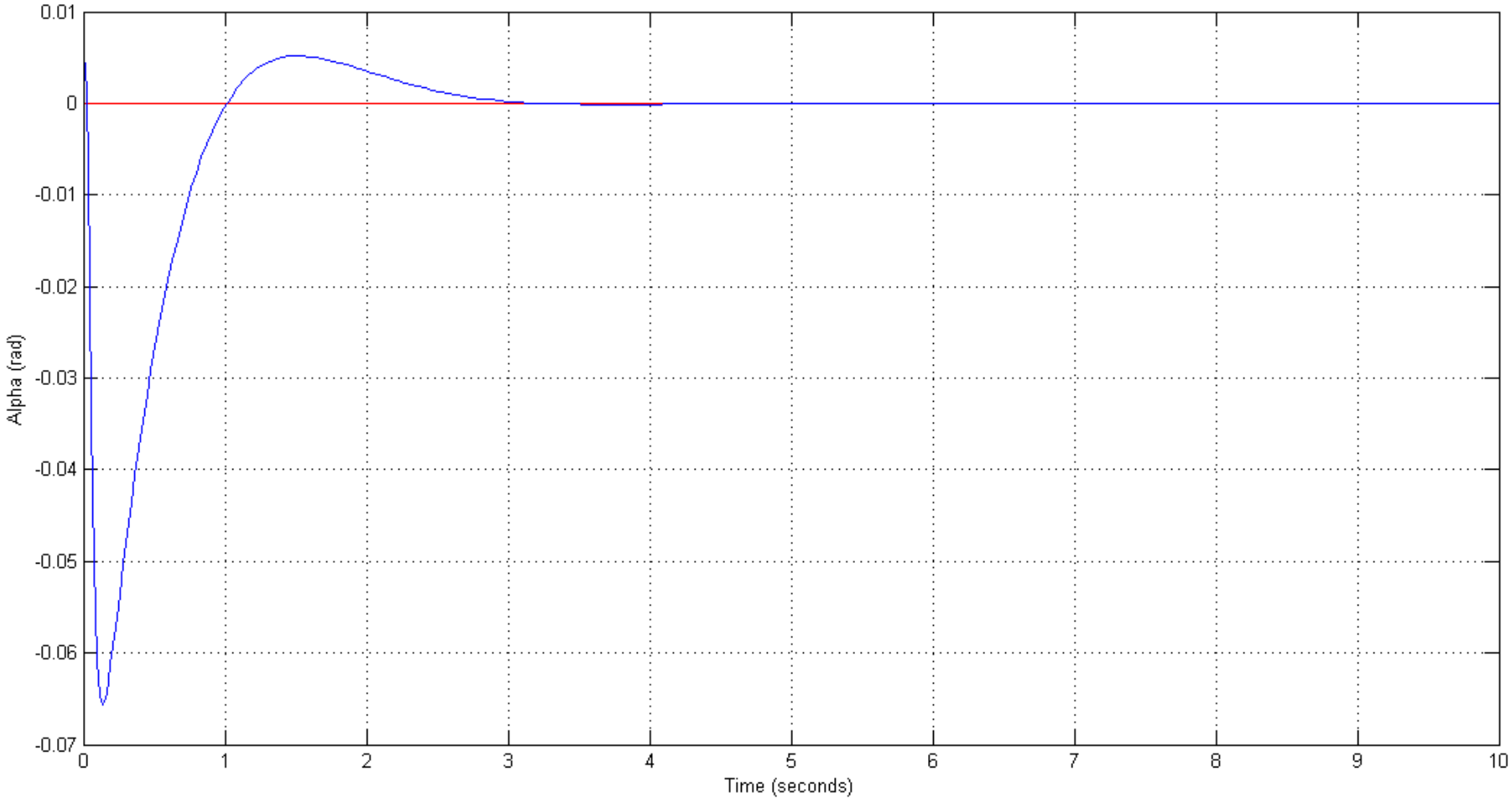
‘ *K* ’ is found to be :

$$K = \begin{bmatrix} -0.3162 & 4.5266 & -0.2427 & 0.4090 \end{bmatrix}$$

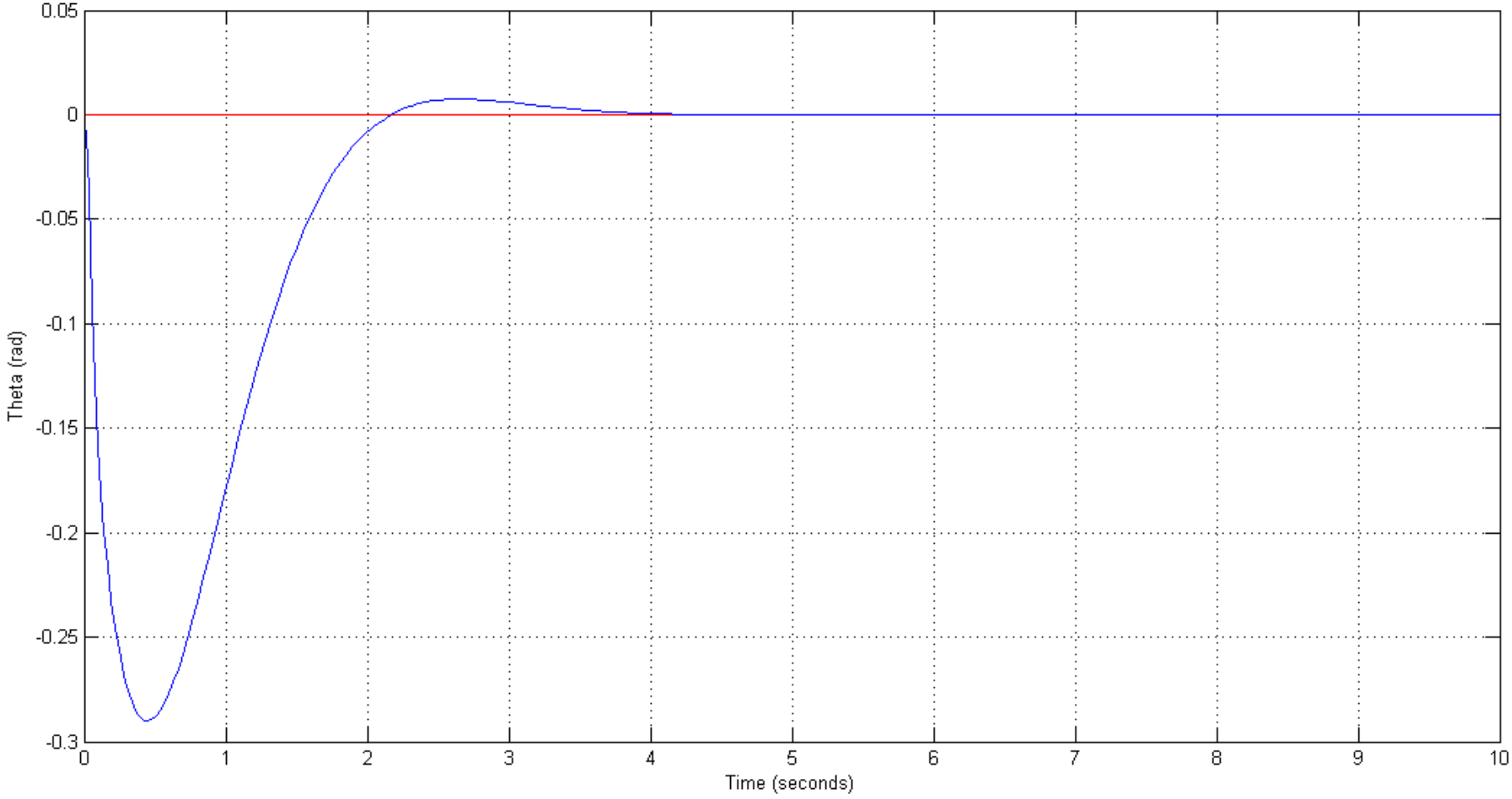
BLOCK DIAGRAM FOR LQR CONTROLLER



Simulation results for ‘α’ using LQR



Simulation results for ‘θ’ using LQR



STATE OBSERVER

Observability

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

O is the Observable Matrix ,
Rank of O is found to be 4 , hence the system is completely observable

STATE OBSERVER

- Full-order state observer: observes all state variables of the system.
- Thus, we define the mathematical model of the observer to be:

$$\begin{aligned}\dot{\tilde{\mathbf{x}}} &= \mathbf{A}\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K_e}(\mathbf{y} - \mathbf{C}\tilde{\mathbf{x}}) \\ &= (\mathbf{A} - \mathbf{K_e}\mathbf{C})\tilde{\mathbf{x}} + \mathbf{B}\mathbf{u} + \mathbf{K_e}\mathbf{y}\end{aligned}$$

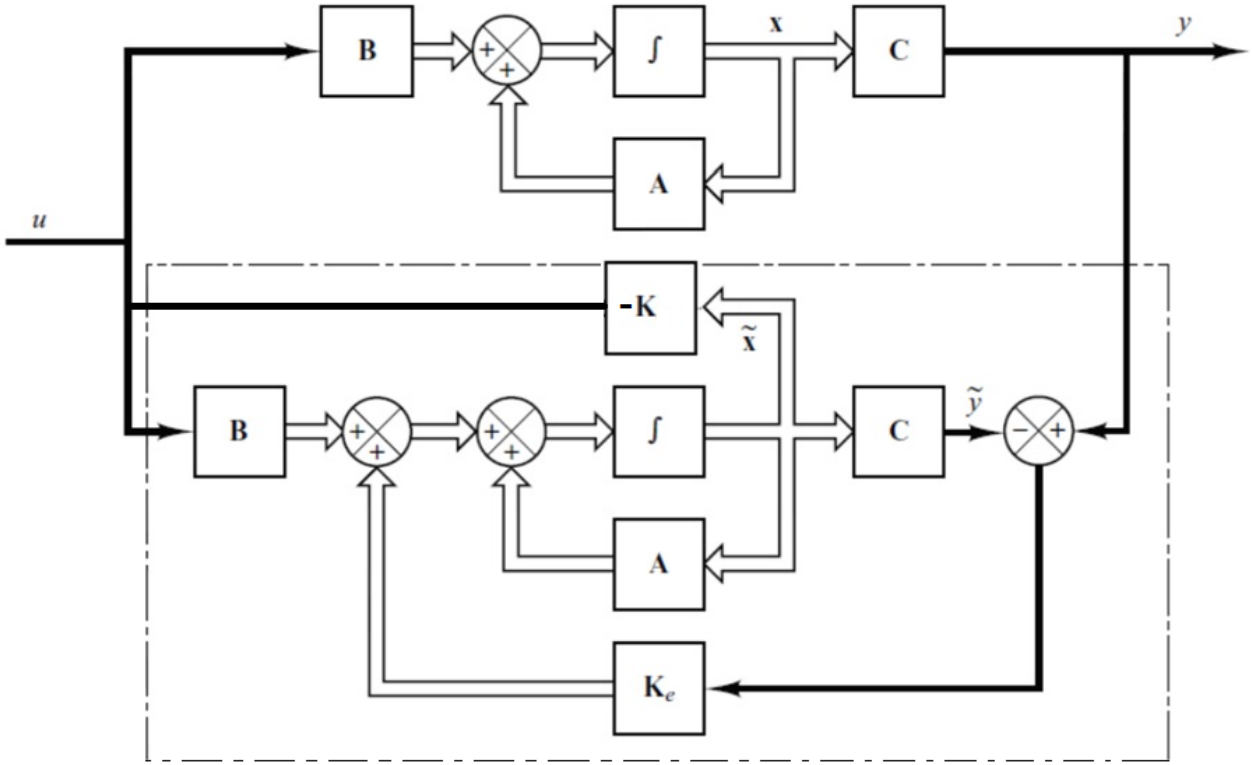
$$\dot{\mathbf{x}} - \dot{\tilde{\mathbf{x}}} = \mathbf{A}\mathbf{x} - \mathbf{A}\tilde{\mathbf{x}} - \mathbf{K_e}(\mathbf{C}\mathbf{x} - \mathbf{C}\tilde{\mathbf{x}})$$
$$= (\mathbf{A} - \mathbf{K_e}\mathbf{C})(\mathbf{x} - \tilde{\mathbf{x}})$$
$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{K_e}\mathbf{C})\mathbf{e}$$

On substituting $u = -\mathbf{K}\tilde{\mathbf{x}}$ in state space equation we get,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} - \mathbf{B}\mathbf{K}\tilde{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}(\mathbf{x} - \tilde{\mathbf{x}})$$
$$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}\mathbf{K}\mathbf{e}$$

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{e}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{K_e}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{e} \end{bmatrix}$$

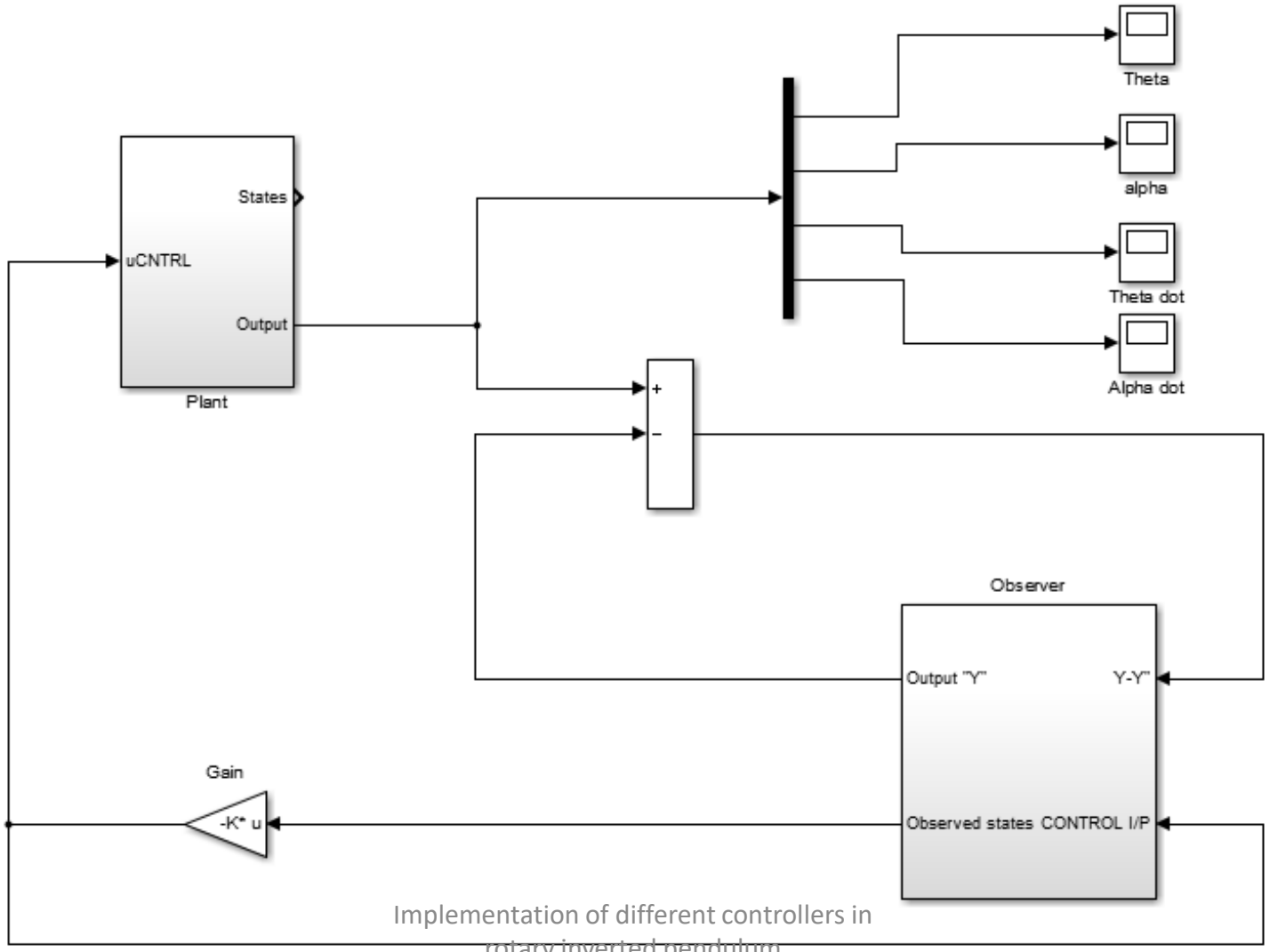
Block Diagram : State Observer method



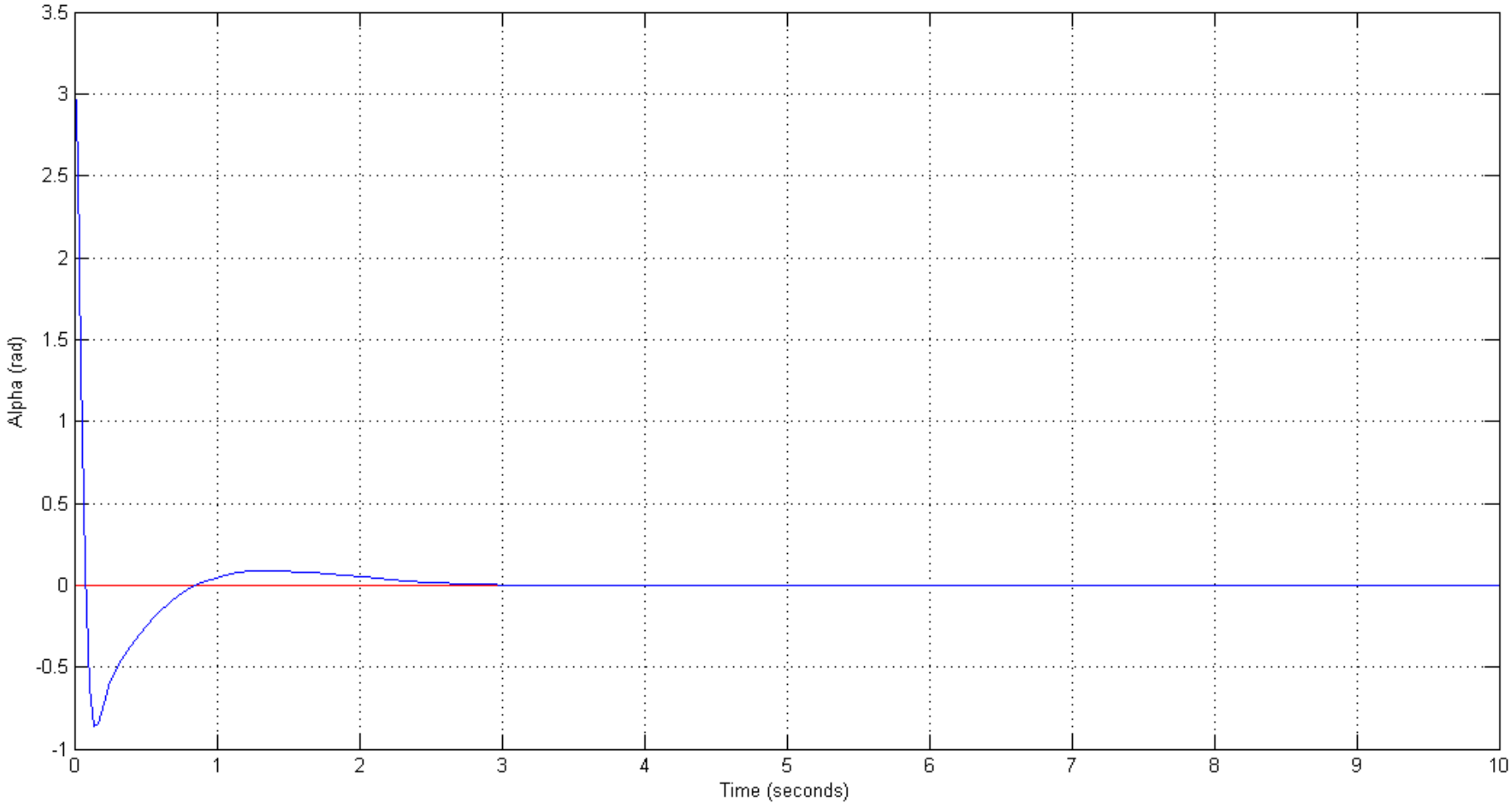
Steps to design State Observer

1. Check observability of the system
2. Assume full-state observer i.e. C = eye(4)
3. Obtain closed-loop gain (from LQR method) and desired closed-loop poles (eigen values of A-BK).
4. Assume observer poles as per requirement.
5. Place observer poles to dual of initial system.
6. Plot the response

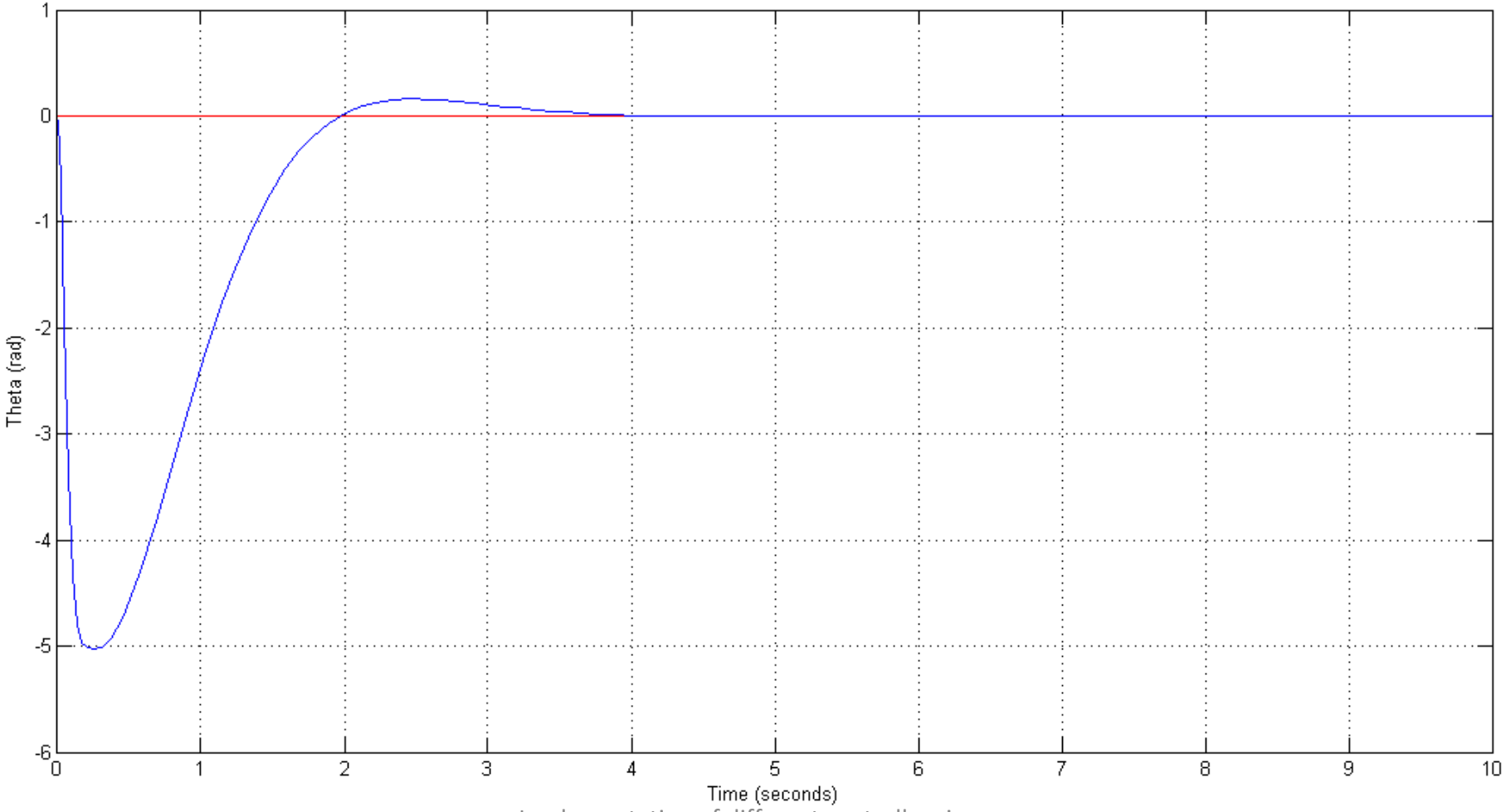
BLOCK DIAGRAM FOR STATE OBSERVER



Simulation results for ‘α’ using Observer



Simulation results for ‘θ’ using Observer



PID Controller

- Desired characteristic equation obtained from LQR method is :

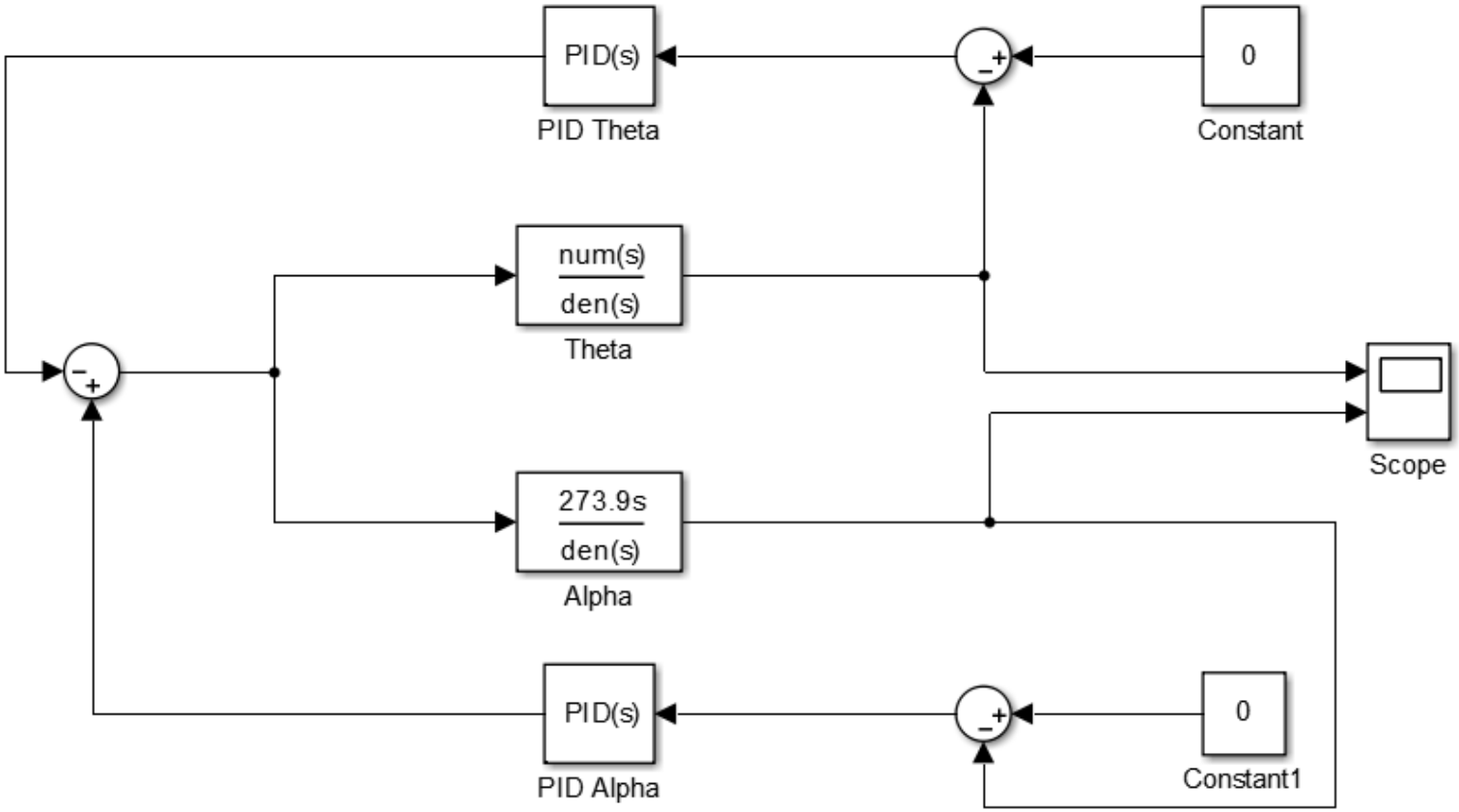
$$s^4 + 42.53s^3 + 1010s^2 + 2840s + 3850 = 0$$

- Using two PID controllers one each for alpha and theta , the characteristic equation obtained is of order 5.
- For comparing both the equations, the above equation is multiplied by (S+10)

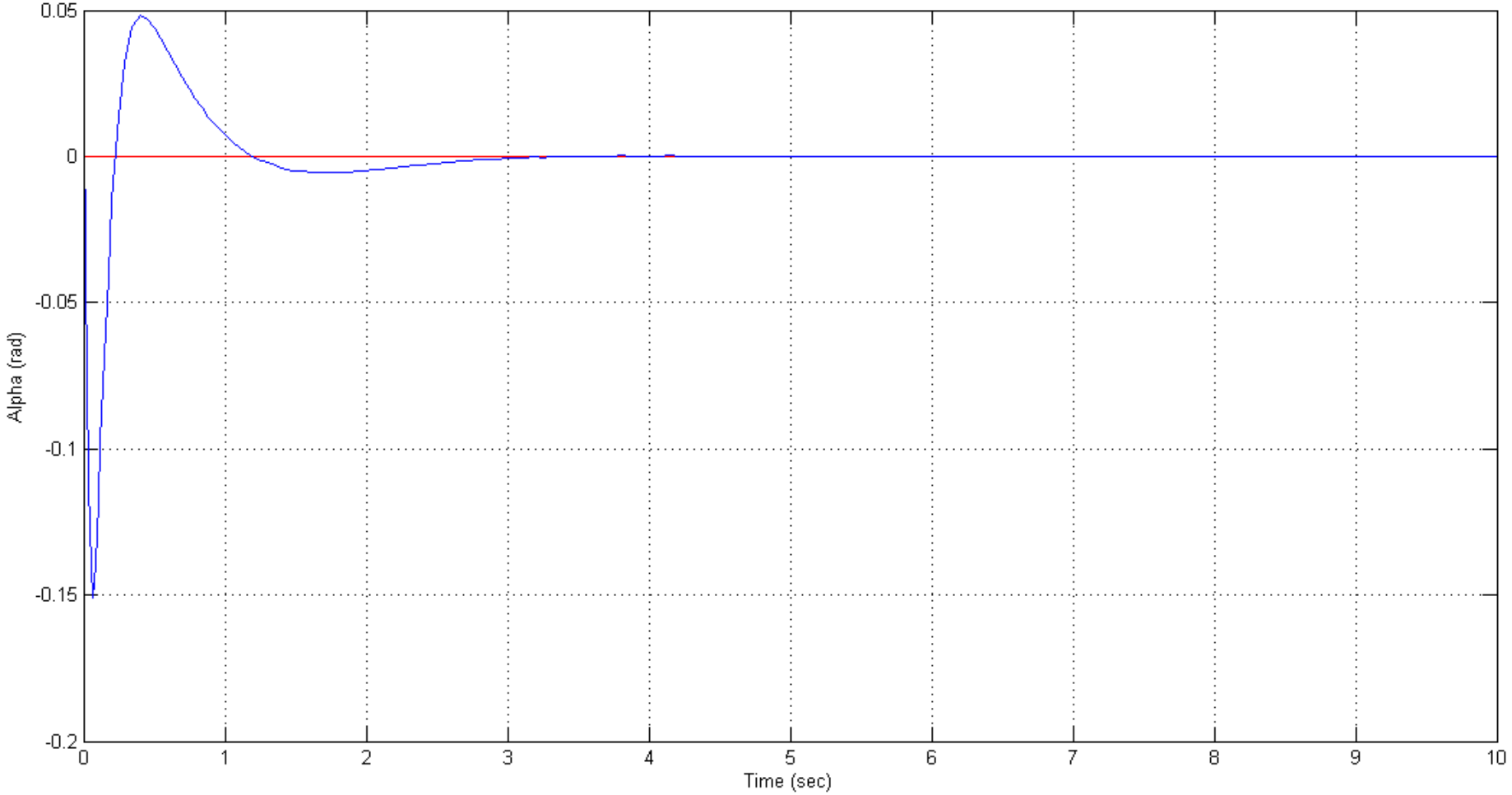
- The Gain Values obtained by comparing both the equations are given by :

Gains	Theta	Alpha
K_p	2.32	5.836
K_d	0	0.303
K_i	3.76	42.785

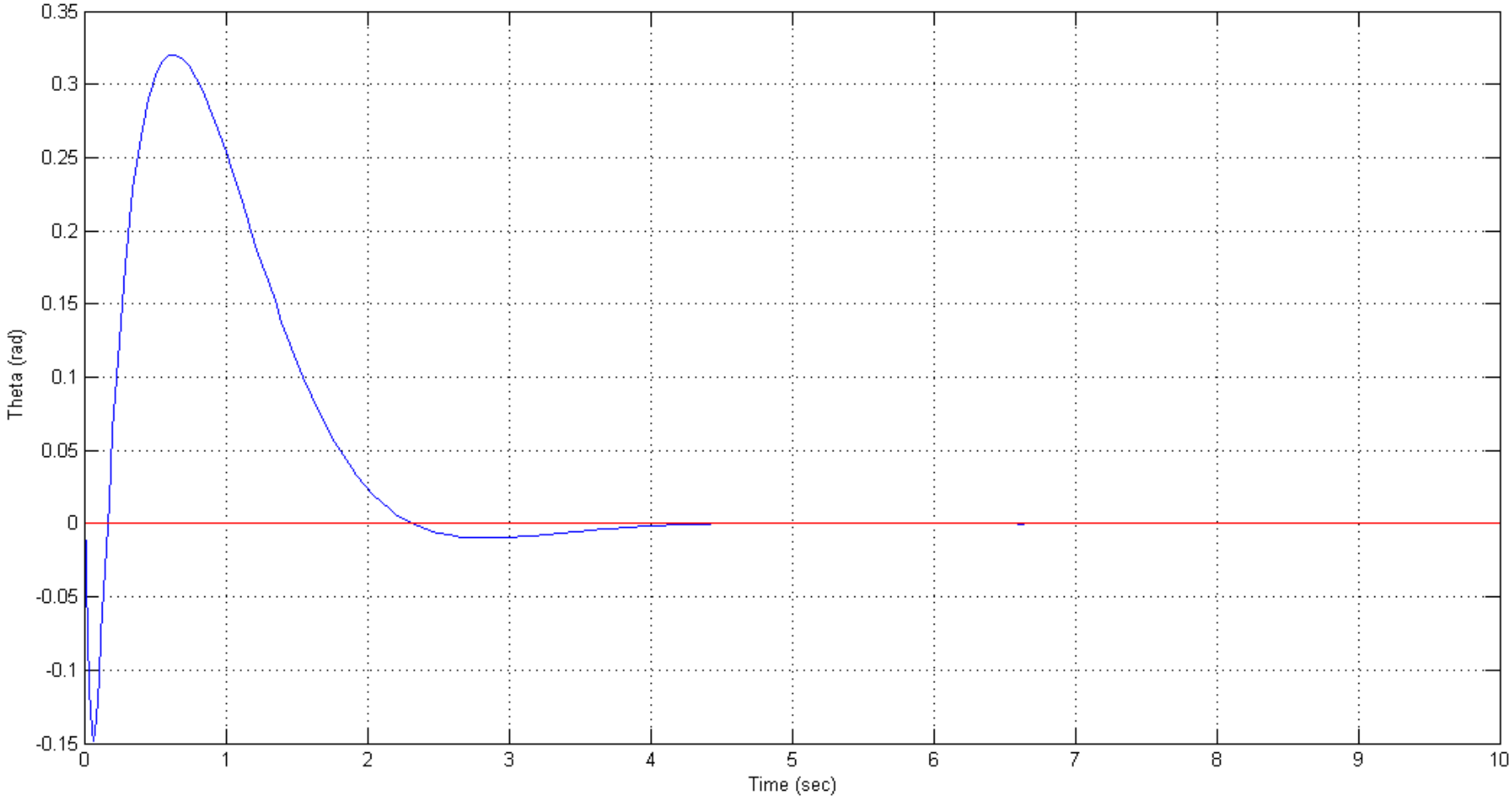
BLOCK DIAGRAM FOR PID CONTROLLER



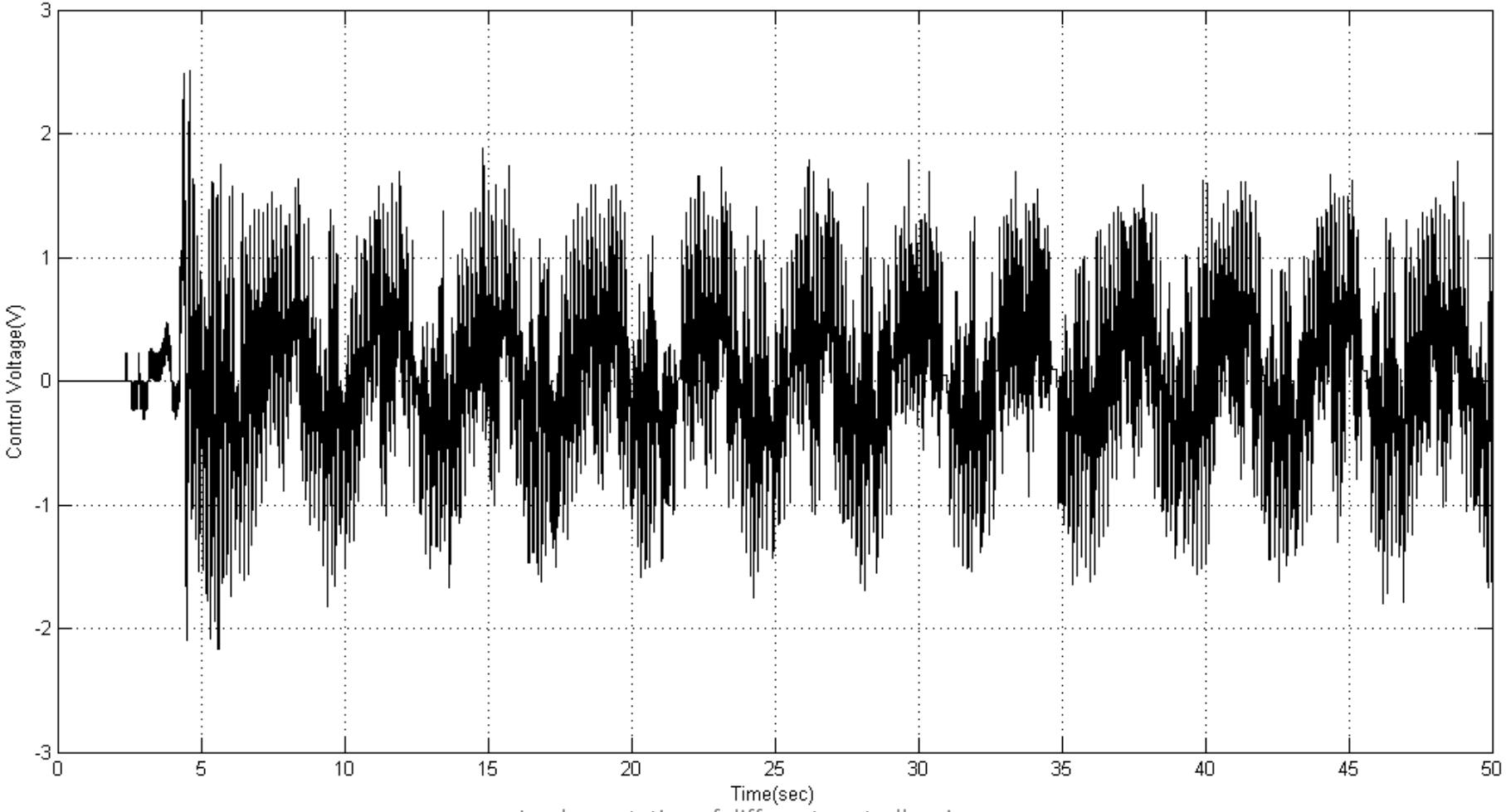
Simulation results for ‘α’ using PID



Simulation results for ‘θ’ using PID

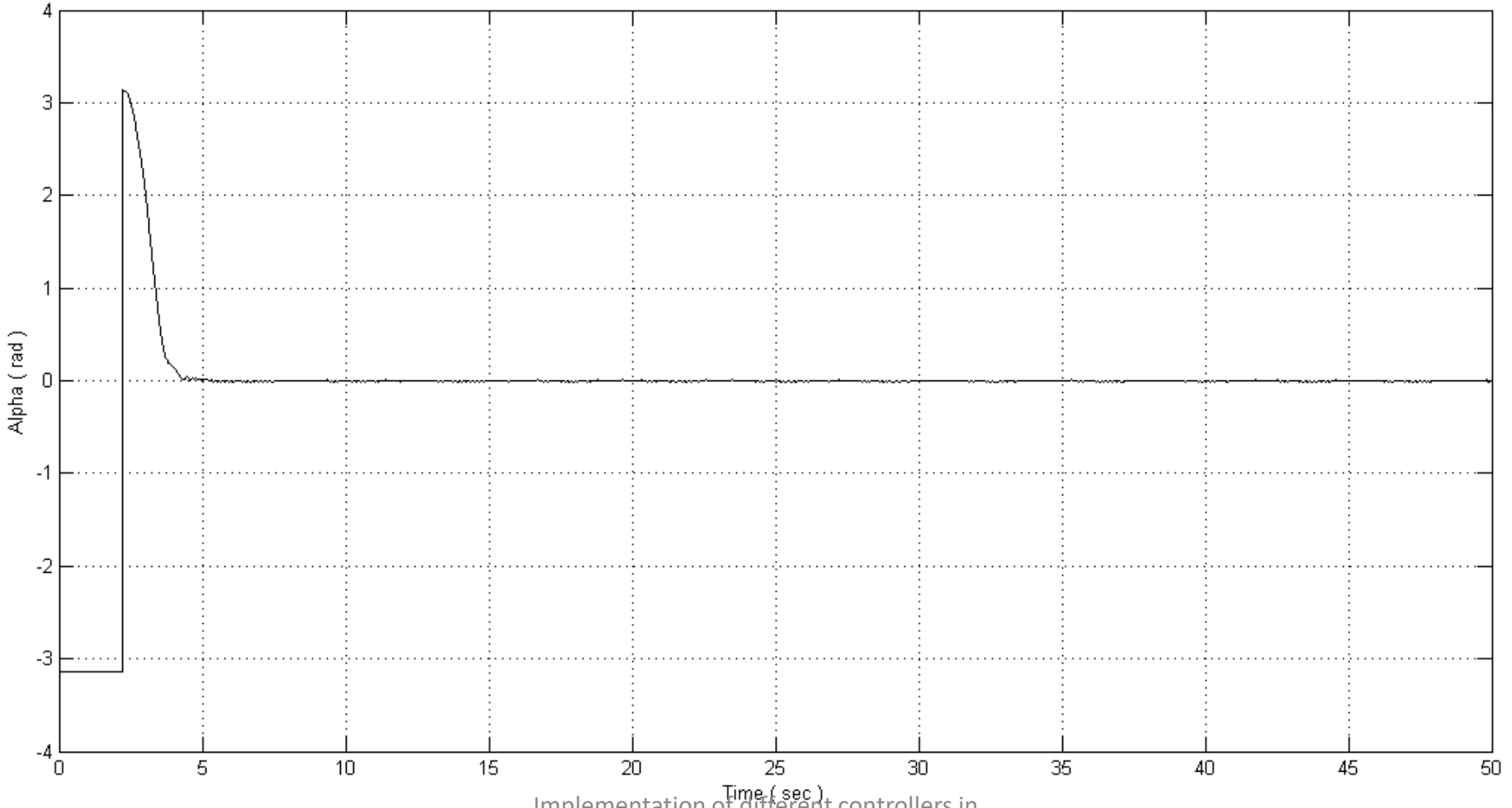


LQR : Control Voltage Vs Time

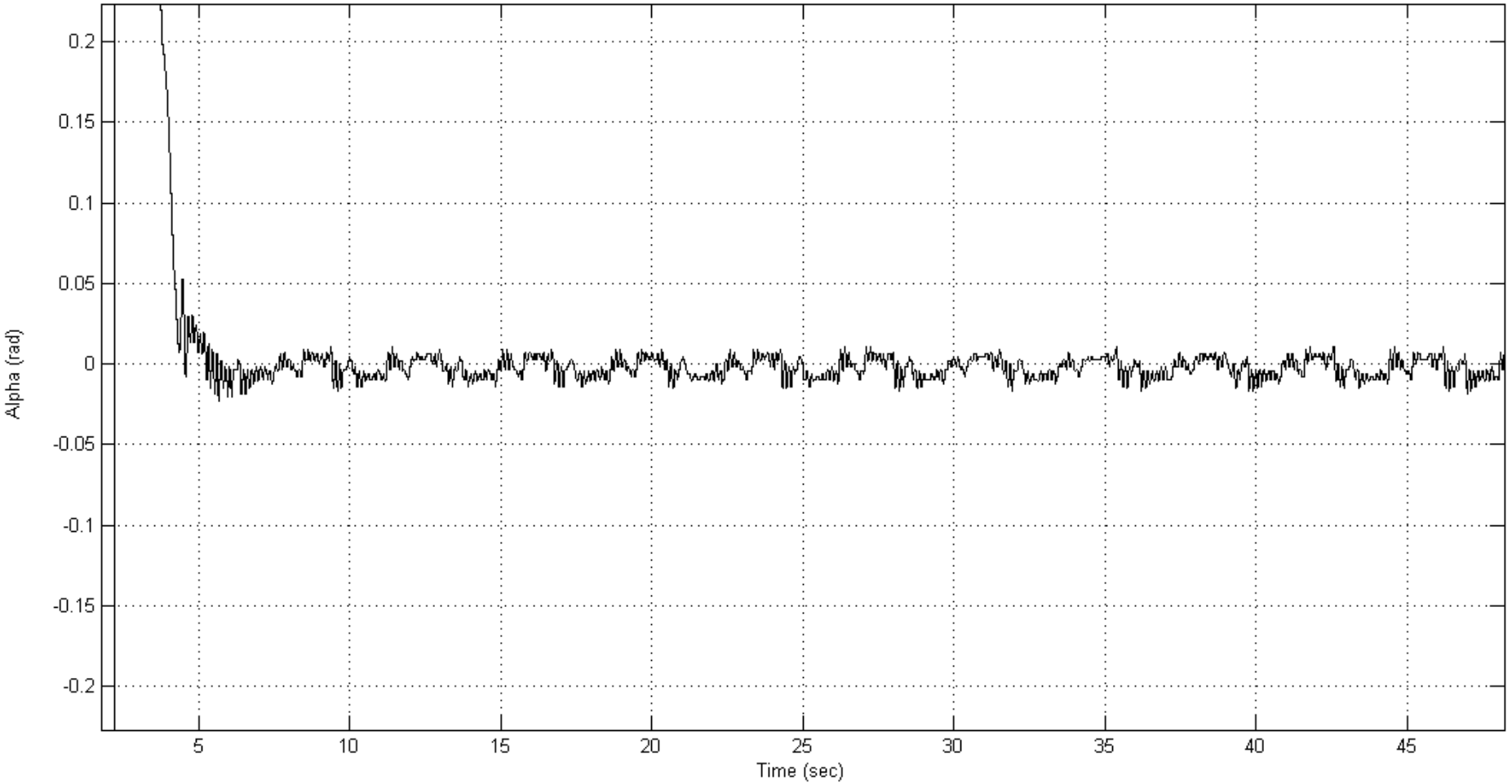


Experimental Results

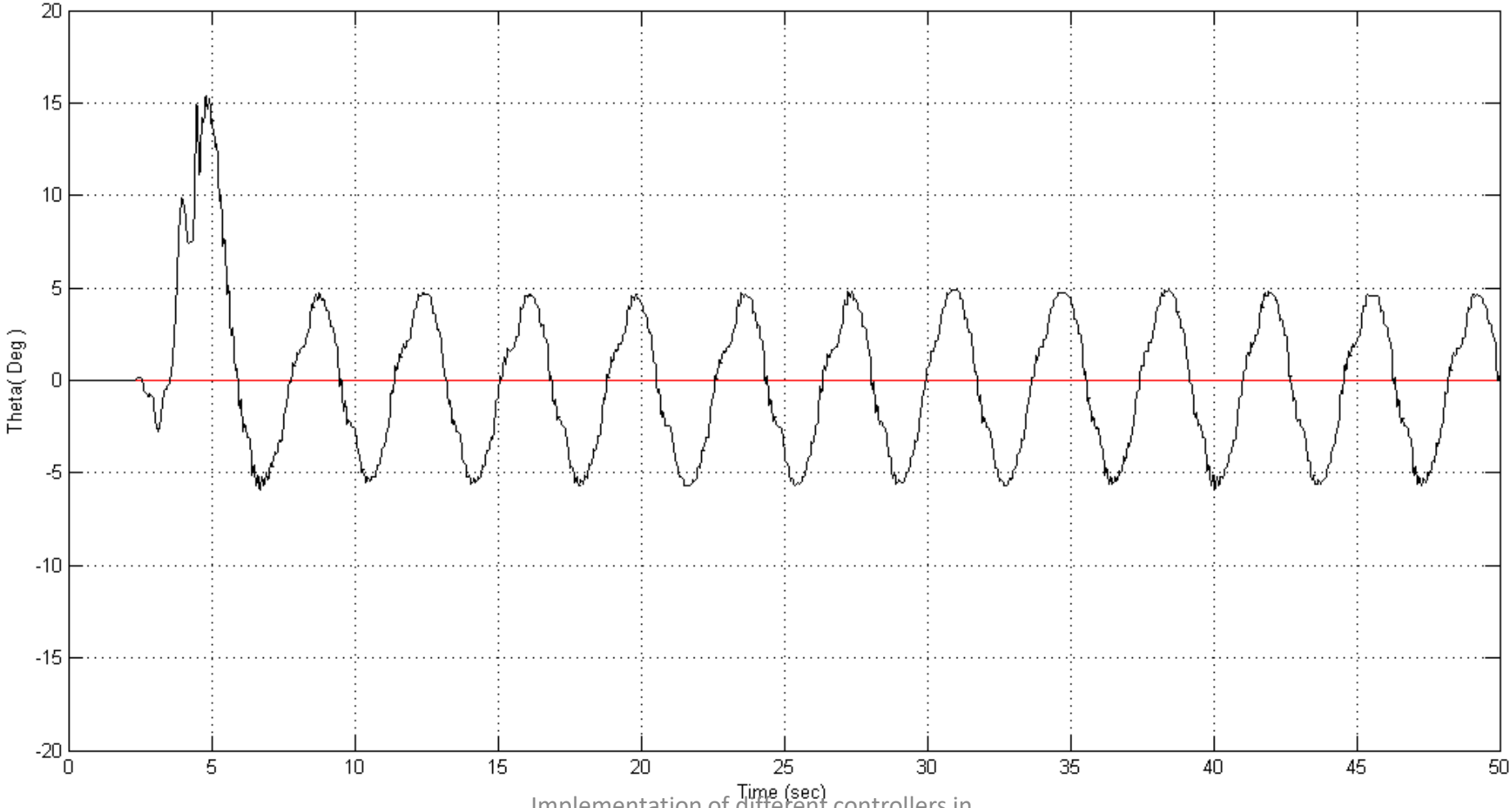
Experimental results for ‘α’ using LQR



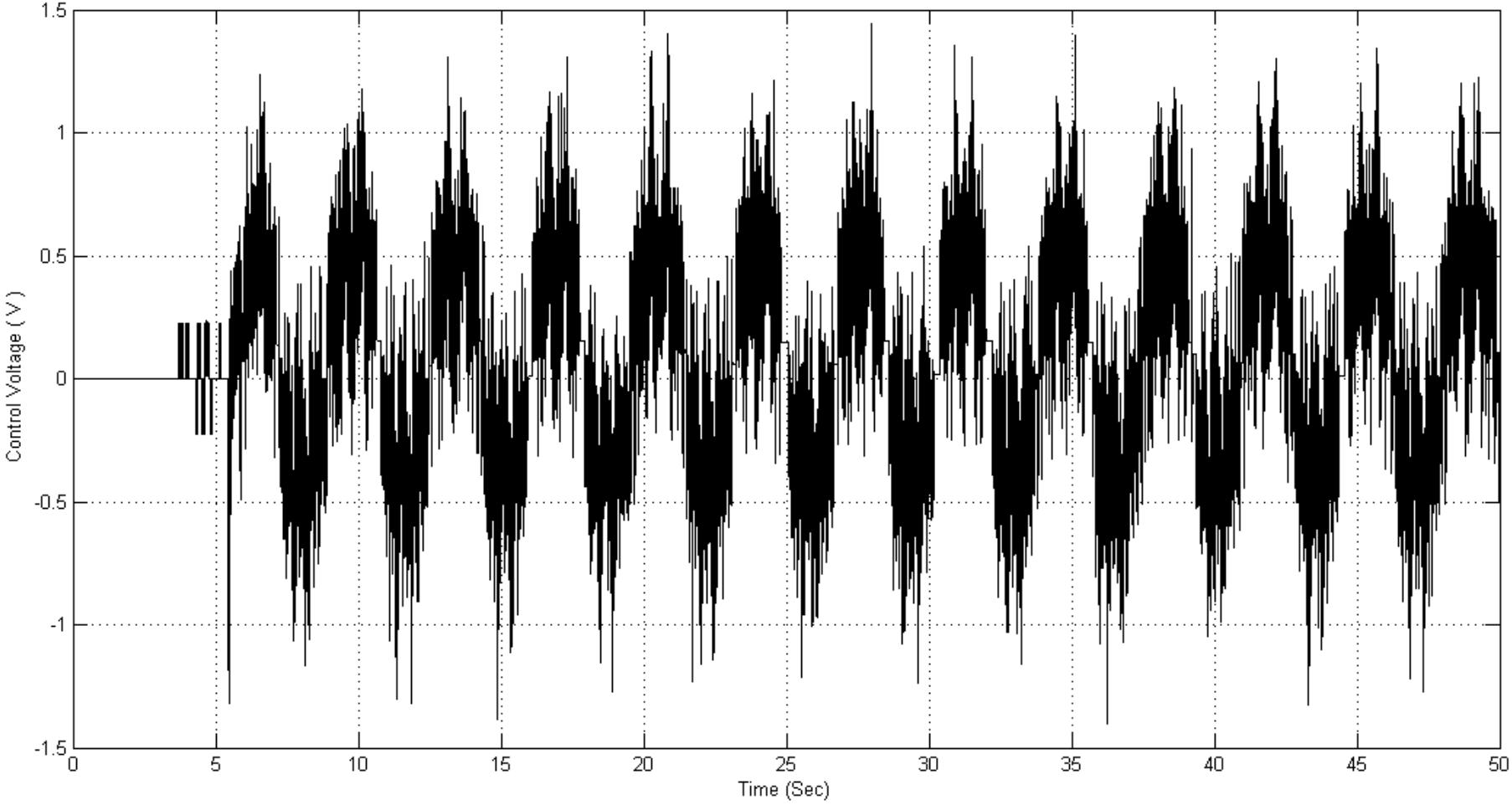
Experimental results for ‘α’ using LQR



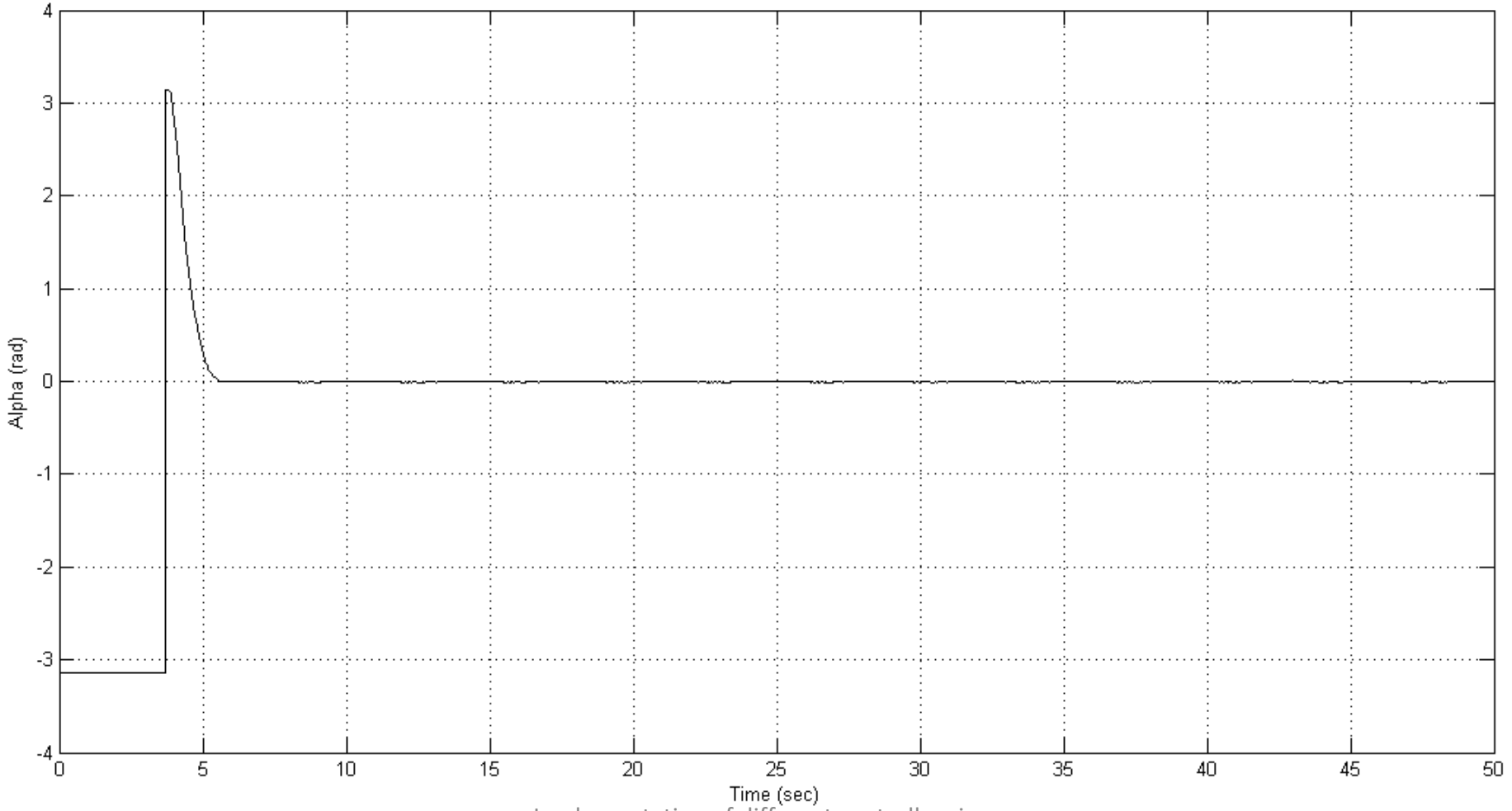
Experimental results for ‘θ’ using LQR



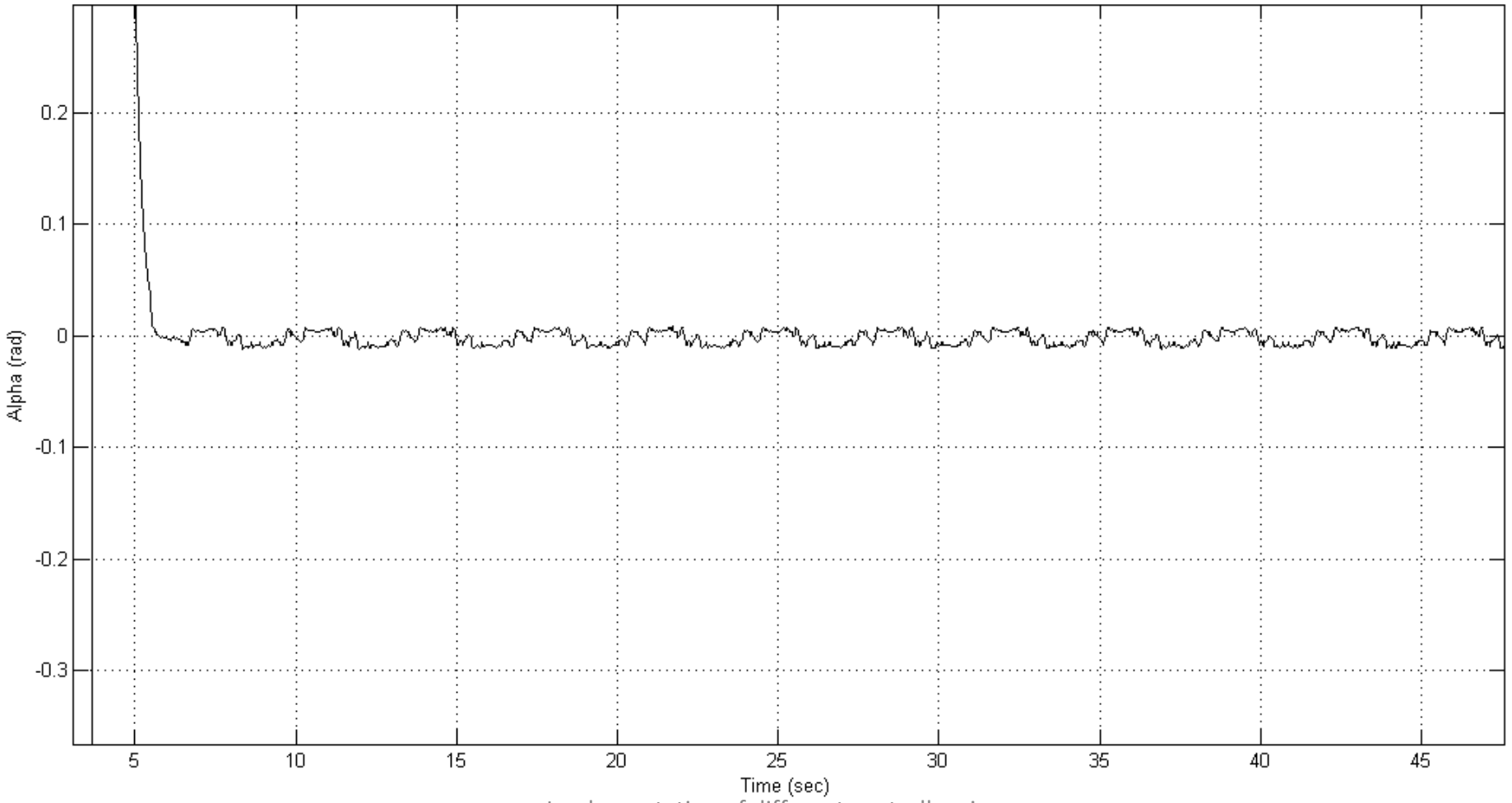
Observer : Control Voltage Vs Time



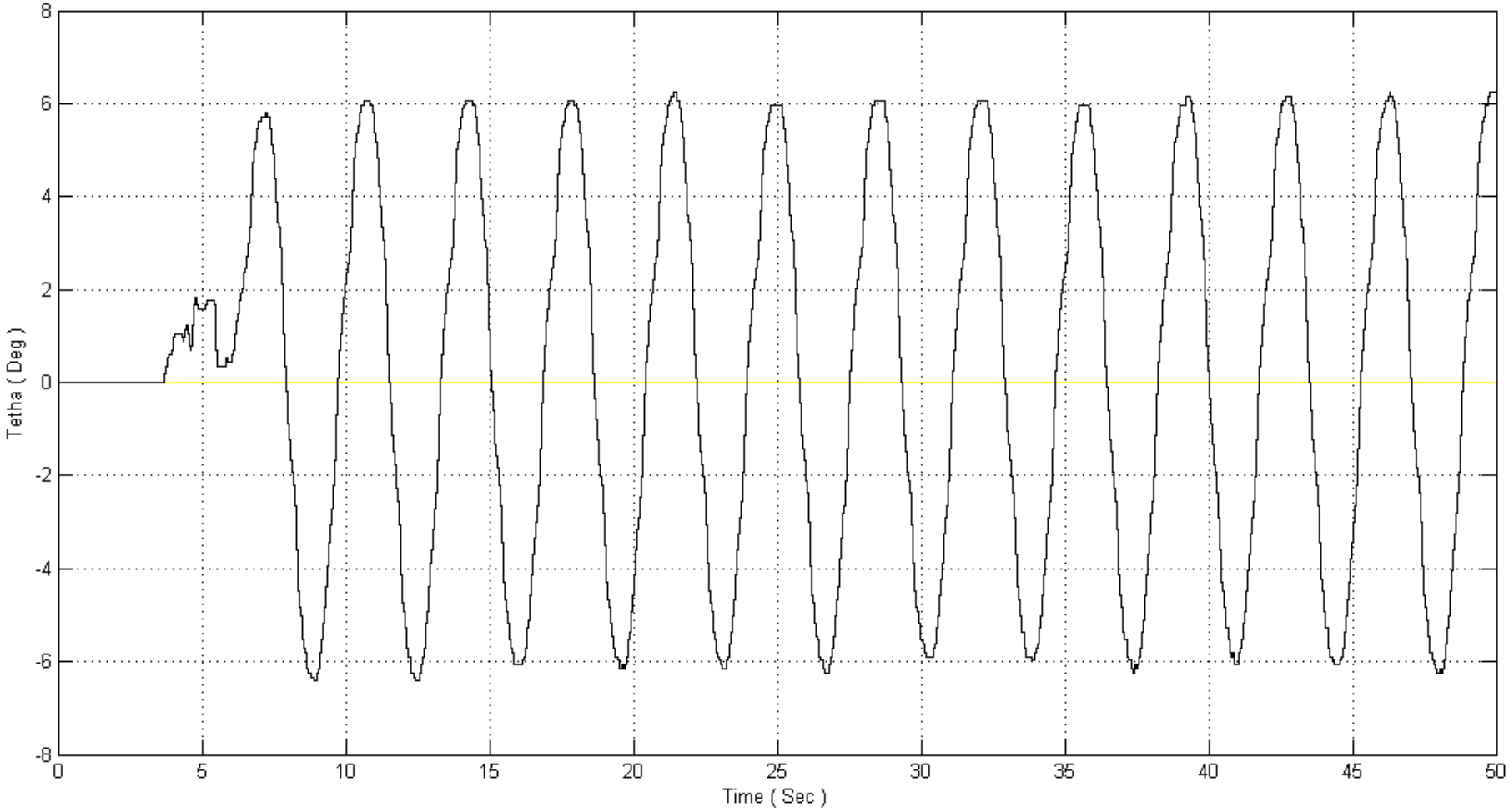
Experimental results for ‘α’ using Observer



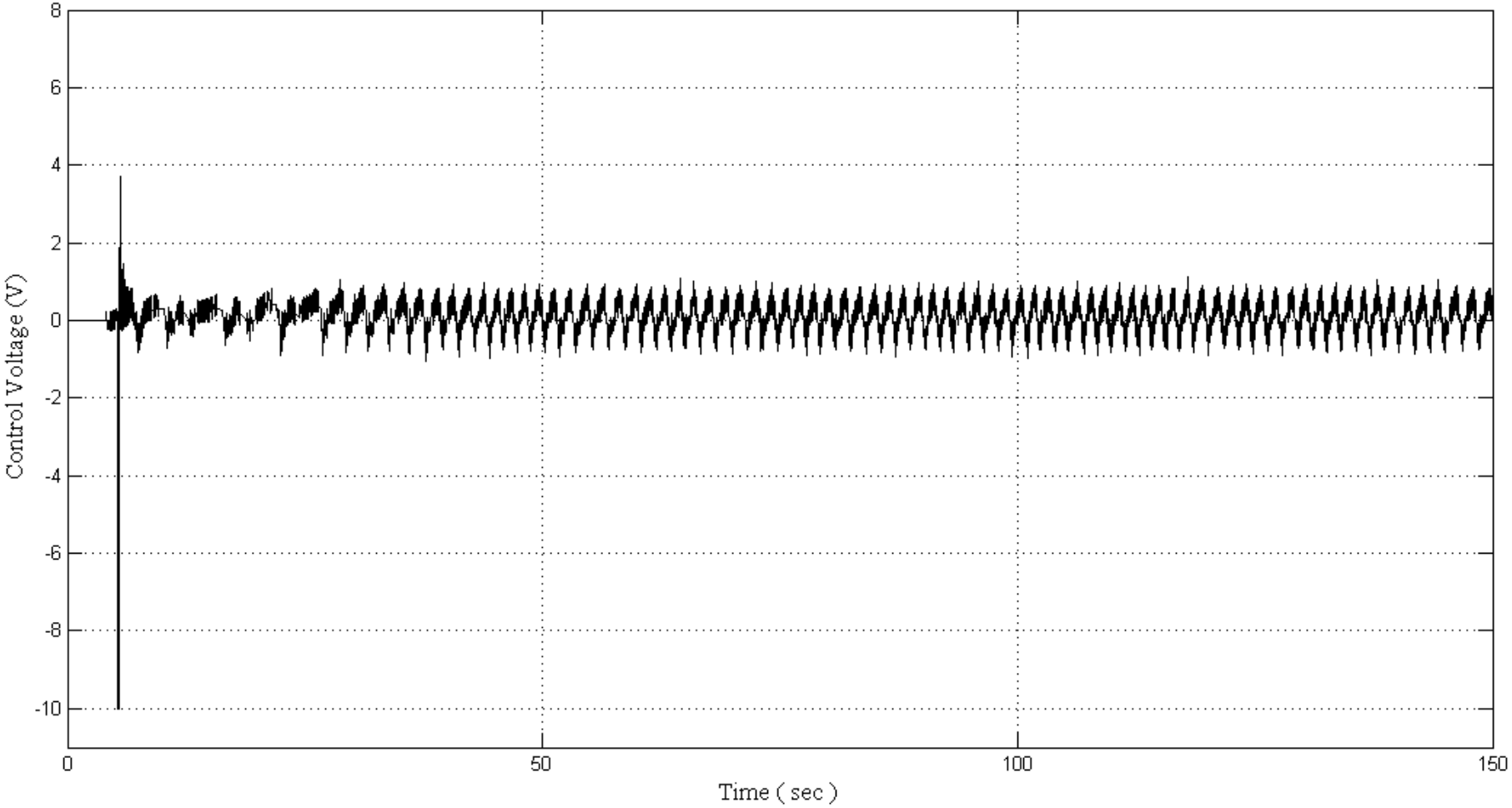
Experimental results for ‘α’ using Observer



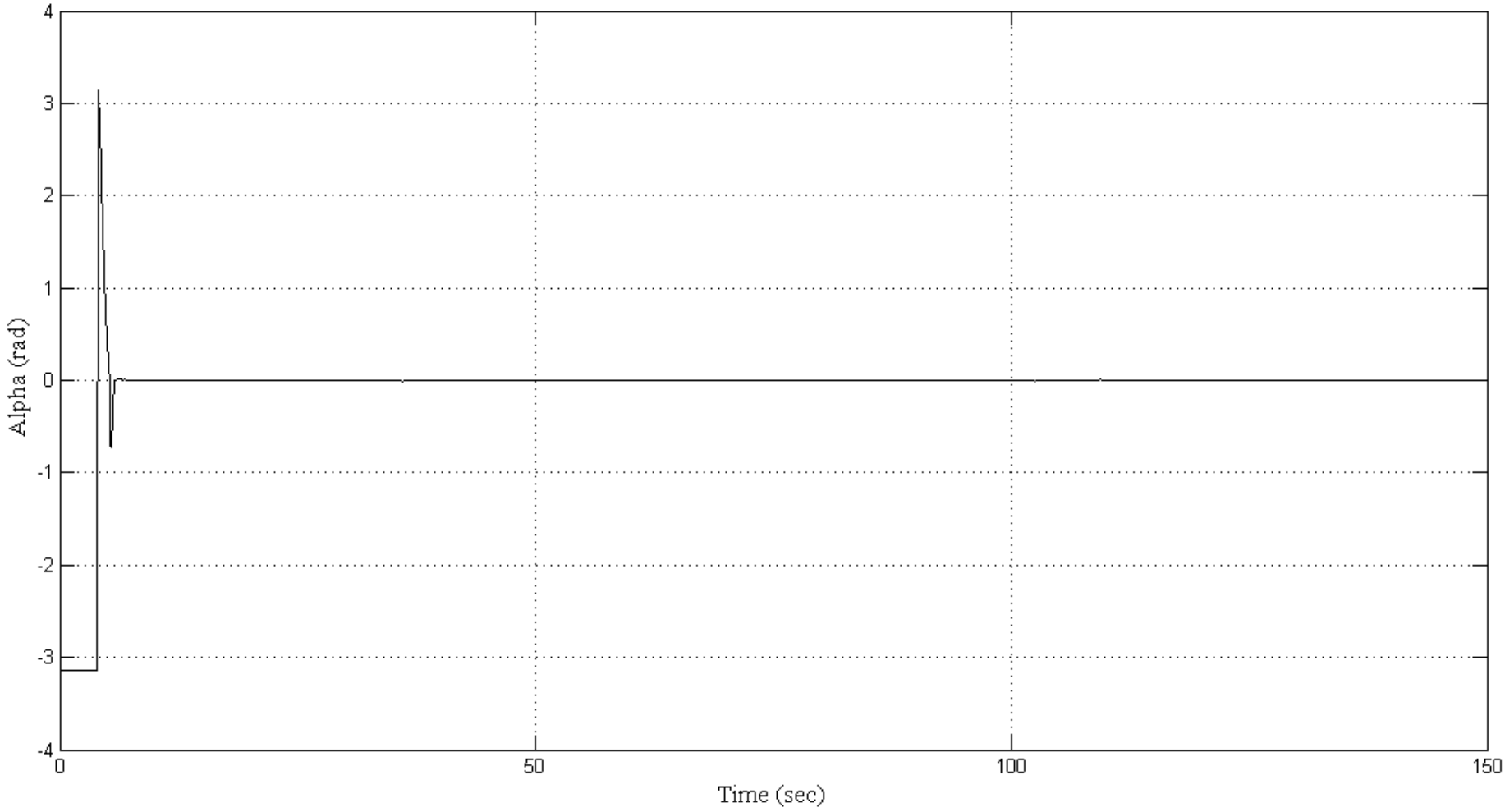
Experimental results for ‘θ’ using Observer



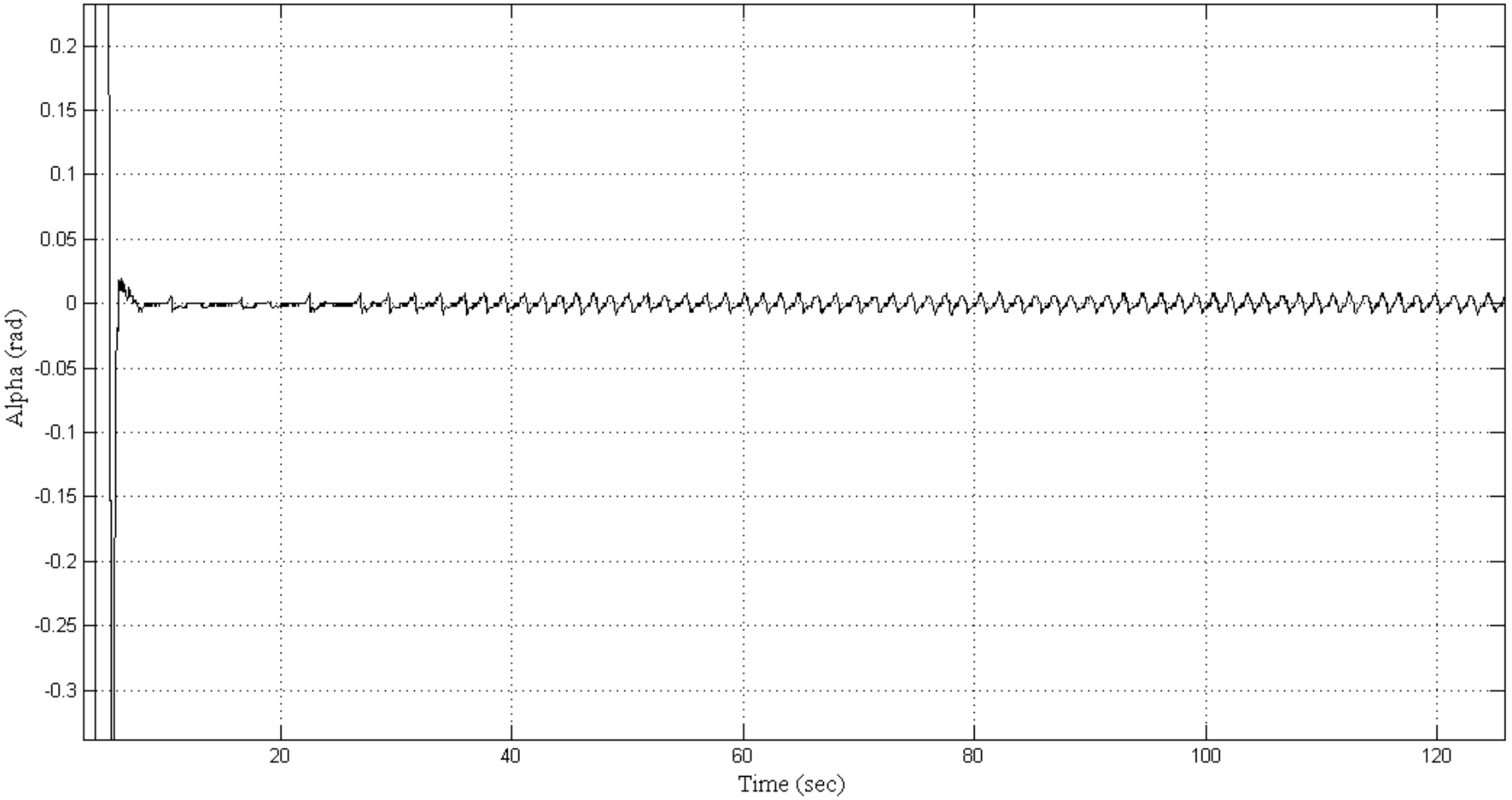
PID : Control Voltage Vs Time



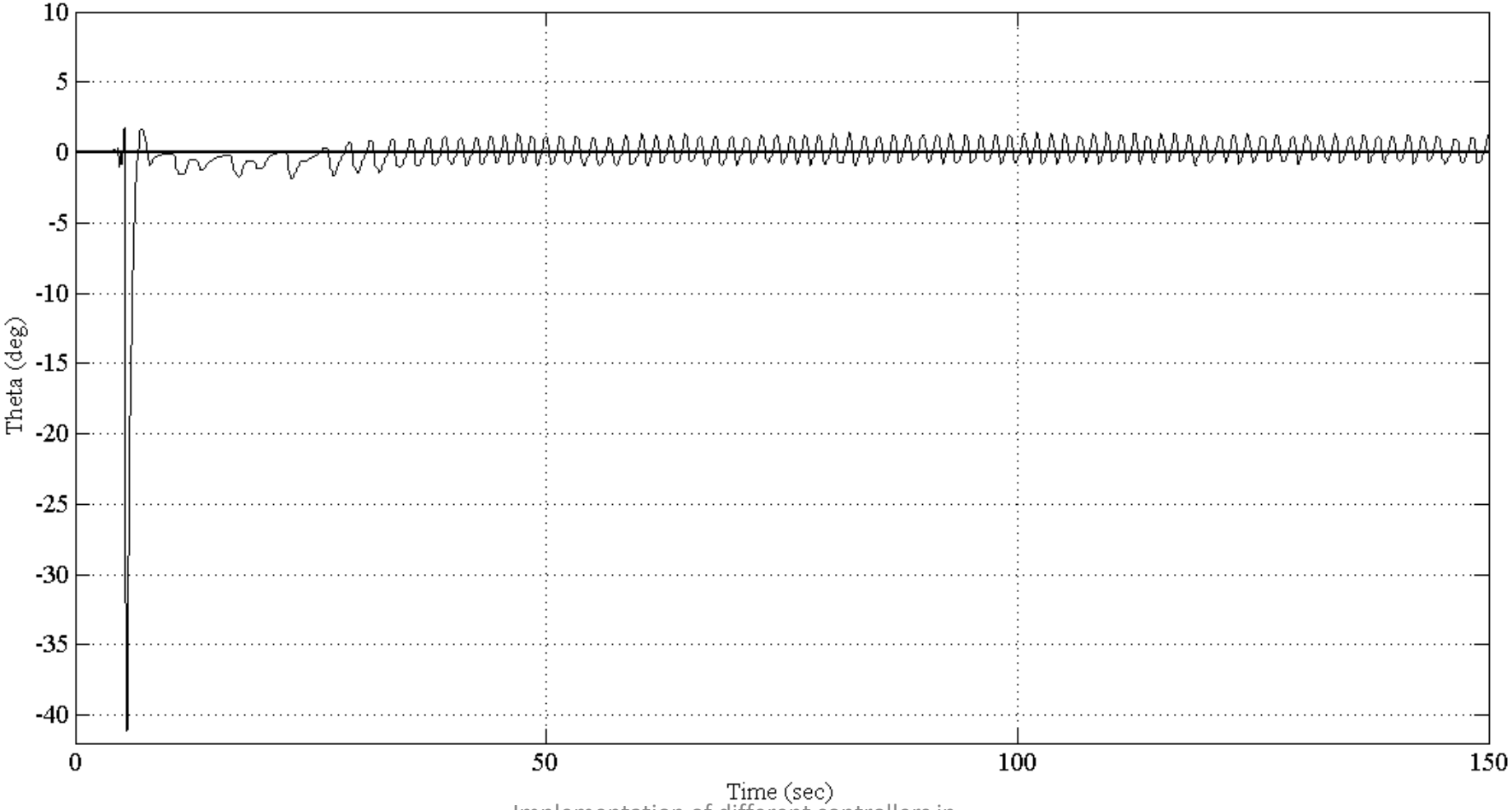
Experimental results for ‘α’ using PID



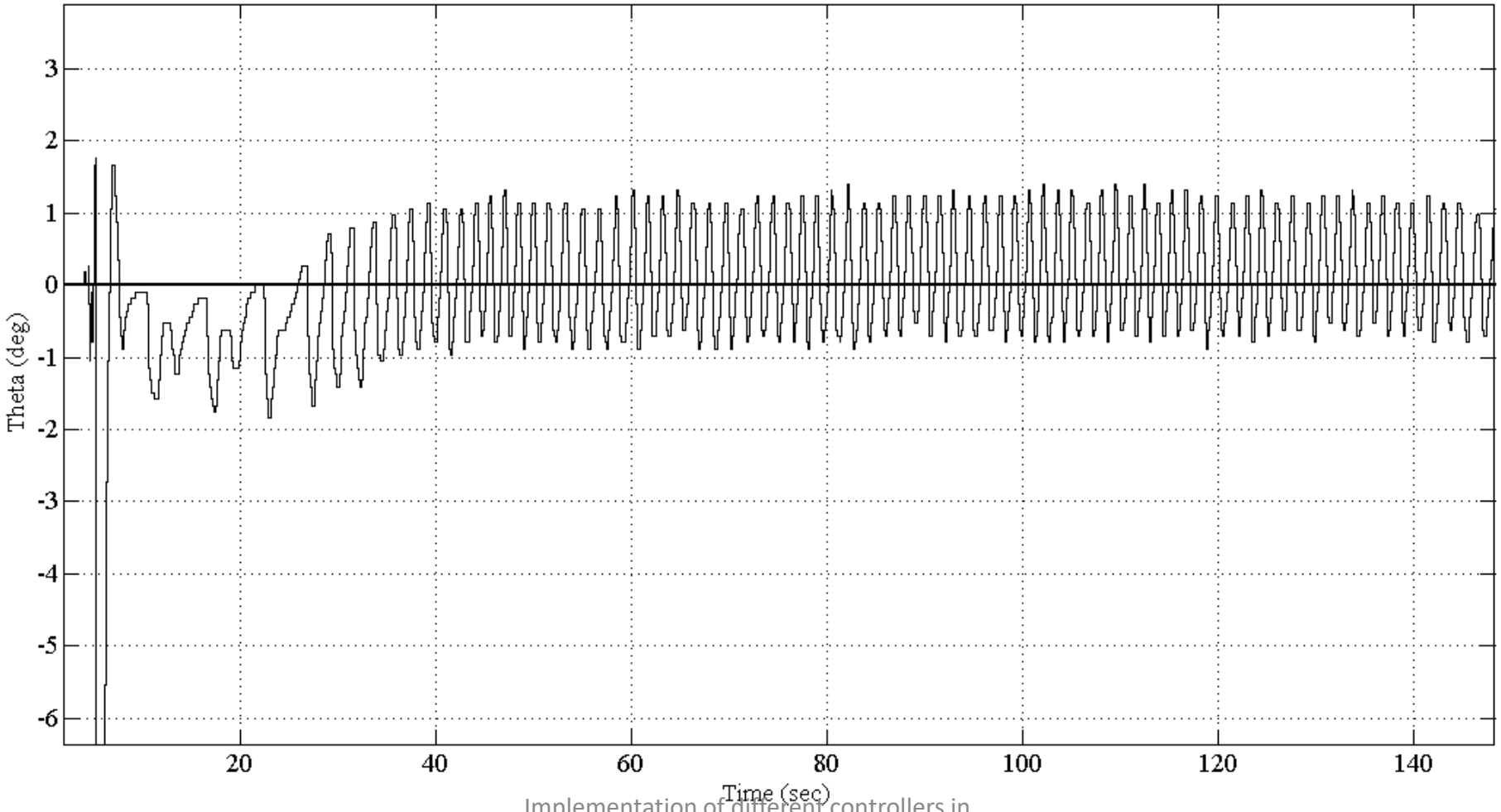
Experimental results for ‘α’ using PID



Experimental results for ‘θ’ using PID



Experimental results for ‘θ’ using PID



Experimental Values

	LQR	Observer	PID
Theta (deg)	-5 to 5	-6 to 6	-1 to 1.5
Alpha (deg)	-1.4 to 0.85	-1.14 to 0.75	-0.57 to 0.57
Control Signal (V)	-1.8V to +1.8V	-1.5V to +1.5V	-1V to +1V

Conclusion

- After implementing the three controller on the Quanser rotary set-up, we come to the following conclusion:
  - Desired characteristic equation is best due to optimization of the system using LQR.
  - State observer – sluggish while implementation.
  - PID Controller – most robust.
  - Control voltage applied to all three controllers is more or less the same.
  - Computation of the PID gain values was more tedious compared to State observer and Pole Placement design.

Conclusion

- Hence, in conclusion, we observe that
  - The LQR technique provides us with the optimal position of the closed loop poles to stabilize the system.
  - The PID tuned to this optimum characteristic equation provides high performance and robustness. This is probably the main reason why PID controllers are common in industrial applications
  - The state observer, although designed using the optimal values, exhibits sluggish behavior and hence, is not a preferred choice of controller in industrial control systems.

THANK YOU!